DP IB Maths: AA HL

1.8 Complex Numbers

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1.8.1 Intro to Complex Numbers

Cartesian Form

What is an imaginary number?

- Up until now, when we have encountered an equation such as $x^2 = -1$ we would have stated that there are "no real solutions"
  - The solutions are $x = \pm\sqrt{-1}$ which are not real numbers
- To solve this issue, mathematicians have defined one of the square roots of negative one as i; an imaginary number
  - $\sqrt{-1} = \mathrm{i}$
  - $\mathrm{i}^2 = -1$
- The square roots of other negative numbers can be found by rewriting them as a multiple of $\sqrt{-1}$
  - using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

What is a complex number?

- Complex numbers have both a real part and an imaginary part
  - For example: $3 + 4\mathrm{i}$
  - The real part is 3 and the imaginary part is 4
    - Note that the imaginary part does not include the 'i'
- Complex numbers are often denoted by z
  - We refer to the real and imaginary parts respectively using $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$
- Two complex numbers are equal if, and only if, both the real and imaginary parts are identical.
  - For example, $3 + 2\mathrm{i}$ and $3 + 3\mathrm{i}$ are **not equal**
- The set of all complex numbers is given the symbol $\mathbb{C}$

What is Cartesian Form?

- There are a number of different forms that complex numbers can be written in
- The form $z = a + b\mathrm{i}$ is known as **Cartesian Form**
  - $a, b \in \mathbb{R}$
  - This is the first form given in the formula booklet
- In general, for $z = a + b\mathrm{i}$
  - Re(z) = a
  - Im(z) = b
- A complex number can be easily represented geometrically when it is in Cartesian Form
- Your GDC may call this **rectangular form**
  - When your GDC is set in rectangular settings it will give answers in Cartesian Form
  - If your GDC is **not** set in a complex mode it will not give any output in complex number form

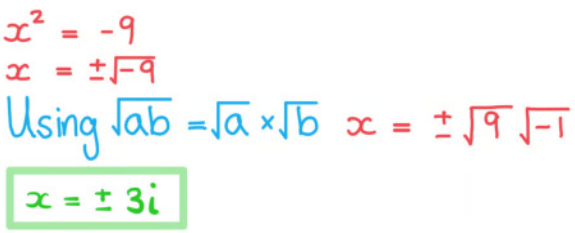
- Make sure you can find the settings for using complex numbers in Cartesian Form and practice inputting problems
- Cartesian form is the easiest form for adding and subtracting complex numbers

Examiner Tip

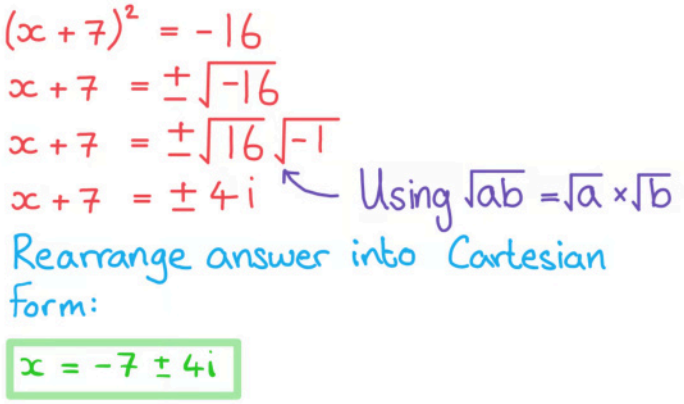
- Remember that complex numbers have both a real part and an imaginary part
  - 1 is purely real (its imaginary part is zero)
  - i is purely imaginary (its real part is zero)
  - 1 + i is a complex number (both the real and imaginary parts are equal to 1)

Worked example

a) Solve the equation $x^2 = -9$



b) Solve the equation $(x + 7)^2 = -16$, giving your answers in Cartesian form.



Complex Addition, Subtraction & Multiplication

How do I add and subtract complex numbers in Cartesian Form?

- Adding and subtracting complex numbers should be done when they are in **Cartesian form**
- When adding and subtracting complex numbers, simplify the real and imaginary parts separately
  - Just like you would when collecting like terms in algebra and surds, or dealing with different components in vectors
  - $(a+b\mathrm{i})+(c+d\mathrm{i})=(a+c)+(b+d)\mathrm{i}$
  - $(a+b\mathrm{i})-(c+d\mathrm{i})=(a-c)+(b-d)\mathrm{i}$

How do I multiply complex numbers in Cartesian Form?

- Complex numbers can be multiplied by a constant in the same way as algebraic expressions:
  - $k(a+b\mathrm{i})=ka+kb\mathrm{i}$
- Multiplying two complex numbers in Cartesian form is done in the same way as multiplying two linear expressions:
  - $(a+b\mathrm{i})(c+d\mathrm{i})=ac+(ad+bc)\mathrm{i}+bd\mathrm{i}^2=ac+(ad+bc)\mathrm{i}-bd$
  - This is a complex number with real part $ac-bd$ and imaginary part $ad+bc$
  - The most important thing when multiplying complex numbers is that
    - $\mathrm{i}^2=-1$
- Your GDC will be able to multiply complex numbers in Cartesian form
  - Practise doing this and use it to check your answers
- It is easy to see that multiplying more than two complex numbers together in Cartesian form becomes a lengthy process prone to errors
  - It is easier to multiply complex numbers when they are in different forms and usually it makes sense to convert them from Cartesian form to either Polar form or Euler's form first
- Sometimes when a question describes multiple complex numbers, the notation z_1, z_2, ... is used to represent each complex number

How do I deal with higher powers of i?

- Because $\mathrm{i}^2=-1$ this can lead to some interesting results for higher powers of i
  - $\mathbf{i}^3=\mathbf{i}^2\times\mathbf{i}=-\mathbf{i}$
  - $\mathbf{i}^4=(\mathbf{i}^2)^2=(-1)^2=1$
  - $\mathbf{i}^5=(\mathbf{i}^2)^2\times\mathbf{i}=\mathbf{i}$
  - $\mathbf{i}^6=(\mathbf{i}^2)^3=(-1)^3=-1$
- We can use this same approach of using i^2 to deal with much higher powers
  - $\mathbf{i}^{23}=(\mathbf{i}^2)^{11}\times\mathbf{i}=(-1)^{11}\times\mathbf{i}=-\mathbf{i}$
  - Just remember that -1 raised to an even power is 1 and raised to an odd power is -1

Examiner Tip

- When revising for your exams, practice using your GDC to check any calculations you do with complex numbers by hand
  - This will speed up using your GDC in rectangular form whilst also giving you lots of practice of carrying out calculations by hand

Worked example

a) Simplify the expression $2(8-6\mathrm{i})-5(3+4\mathrm{i})$.

Expand the brackets

$2(8-6\mathrm{i})-5(3+4\mathrm{i}) = 16-12\mathrm{i}-15-20\mathrm{i}$

Collect the real and imaginary parts

$16-15-12\mathrm{i}-20\mathrm{i}$

Simplify

$1-32\mathrm{i}$

b) Given two complex numbers $z_1 = 3+4\mathrm{i}$ and $z_2 = 6+7\mathrm{i}$, find $z_1 \times z_2$.

Expand the brackets

$(3+4\mathrm{i})(6+7\mathrm{i}) = 18+21\mathrm{i}+24\mathrm{i}+28\mathrm{i}^2$

$= 18+21\mathrm{i}+24\mathrm{i}+(28)(-1)$ ← Using $\mathrm{i}^2 = -1$

Collect the real and imaginary parts

$18+21\mathrm{i}+24\mathrm{i}-28 = 18-28+(21+24)\mathrm{i}$

Simplify

$-10+45\mathrm{i}$

Complex Conjugation & Division

When **dividing** complex numbers, the **complex conjugate** is used to change the denominator to a real number.

What is a complex conjugate?

- For a given complex number $z = a + b\mathrm{i}$, **the complex conjugate of z is denoted as z^***, where $z^* = a - b\mathrm{i}$
- If $z = a - b\mathrm{i}$ then $z^* = a + b\mathrm{i}$
- You will find that:
  - $z + z^*$ is always real because $(a + b\mathrm{i}) + (a - b\mathrm{i}) = 2a$
    - For example: $(6 + 5\mathrm{i}) + (6 - 5\mathrm{i}) = 6 + 6 + 5\mathrm{i} - 5\mathrm{i} = 12$
  - $z - z^*$ is always imaginary because $(a + b\mathrm{i}) - (a - b\mathrm{i}) = 2b\mathrm{i}$
    - For example: $(6 + 5\mathrm{i}) - (6 - 5\mathrm{i}) = 6 - 6 + 5\mathrm{i} - (-5\mathrm{i}) = 10\mathrm{i}$
  - $z \times z^*$ is always real because $(a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + ab\mathrm{i} - ab\mathrm{i} - b^2\mathrm{i}^2 = a^2 + b^2$ (as $\mathrm{i}^2 = -1$)
    - For example: $(6 + 5\mathrm{i})(6 - 5\mathrm{i}) = 36 + 30\mathrm{i} - 30\mathrm{i} - 25\mathrm{i}^2 = 36 - 25(-1) = 61$

How do I divide complex numbers?

- To divide two complex numbers:
  - STEP 1: Express the calculation in the form of a fraction
  - STEP 2: Multiply **the top and bottom by the conjugate of the denominator**:
    - $\dfrac{a + b\mathrm{i}}{c + d\mathrm{i}} = \dfrac{a + b\mathrm{i}}{c + d\mathrm{i}} \times \dfrac{c - d\mathrm{i}}{c - d\mathrm{i}}$
    - This ensures we are multiplying by 1; so not affecting the overall value
  - STEP 3: Multiply out and simplify your answer
    - This should have a real number as the denominator
  - STEP 4: Write your answer in Cartesian form as two terms, simplifying each term if needed
    - OR convert into the required form if needed
- Your GDC will be able to divide two complex numbers in Cartesian form
  - Practise doing this and use it to check your answers if you can

Examiner Tip

- We can speed up the process for finding zz^* by using the basic pattern of $(x+a)(x-a)=x^2-a^2$
- We can apply this to complex numbers: $(a+bi)(a-bi)=a^2-b^2i^2=a^2+b^2$ (using the fact that $i^2=-1$)
  - So $3+4i$ multiplied by its conjugate would be $3^2+4^2=25$

Worked example

Find the value of $(1+7i)\div(3-i)$.

Rewrite as a fraction: $\frac{1+7i}{3-i}$ ← complex conjugate of $3-i$ is $3+i$

Multiply top and bottom of the fraction by the complex conjugate of the denominator.

$$\frac{1+7i}{3-i}\times\frac{3+i}{3+i}=\frac{(1+7i)(3+i)}{(3-i)(3+i)}$$

$$=\frac{3+i+21i+7i^2}{9+3i-3i-i^2}\quad (7i^2=-1)$$

↑ the imaginary parts eliminate each other

$$=\frac{3+22i+(-7)}{9-(-1)}$$

Simplify $=-\frac{4+22i}{10}$

Write in Cartesian form $=-\frac{4}{10}+\frac{22}{10}i$

$-\frac{2}{5}+\frac{11}{5}i$ Simplify final answer.

1.8.2 Modulus & Argument

Modulus & Argument

How do I find the modulus of a complex number?

- The modulus of a complex number is its **distance** from the origin when plotted on an Argand diagram
- The modulus of z is written $|z|$
- If $z = x + \mathrm{i}y$, then we can use **Pythagoras** to show...
  - $|z| = \sqrt{x^2 + y^2}$
- A modulus is **never negative**

What features should I know about the modulus of a complex number?

- the modulus is related to the complex **conjugate** by...
  - $zz^* = z^*z = |z|^2$
  - This is because $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$
- In general, $|z_1 + z_2| \neq |z_1| + |z_2|$
  - e.g. both $z_1 = 3 + 4\mathrm{i}$ and $z_2 = -3 + 4\mathrm{i}$ have a modulus of 5, but $z_1 + z_2$ simplifies to 8**i** which has a modulus of 8

How do I find the argument of a complex number?

- The argument of a complex number is the **angle** that it makes on an **Argand diagram**
  - The angle must be taken from the **positive real axis**
  - The angle must be in a **counter-clockwise** direction
- Arguments are measured in **radians**
  - They can be given exact in terms of π
- The argument of z is written $\arg z$
- Arguments can be calculated using right-angled **trigonometry**
  - This involves using the tan ratio plus a sketch to decide whether it is positive/negative and acute/obtuse

What features should I know about the argument of a complex number?

- Arguments are usually given in the range $-\pi < \arg z \leq \pi$
  - Negative arguments are for complex numbers in the third and fourth quadrants
  - Occasionally you could be asked to give arguments in the range $0 < \arg z \leq 2\pi$
    - The question will make it clear which range to use
- The argument of zero, $\arg 0$ is undefined (no angle can be drawn)

What are the rules for moduli and arguments under multiplication and division?

- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **moduli** are also **multiplied**

- $|z_1 z_2| = |z_1||z_2|$

- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **moduli** are also **divided**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **arguments** are **added**
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **arguments** are **subtracted**

Examiner Tip

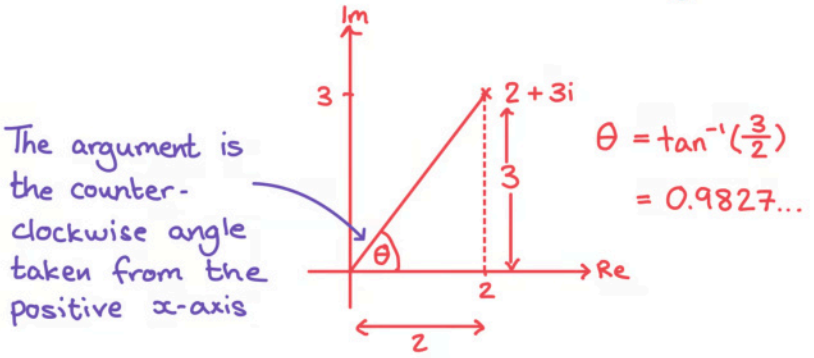
- Always draw a quick sketch to help you see what quadrant the complex number lies in when working out an argument
- Look for the range of values within which you should give your argument
  - If it is $-\pi < \arg z \leq \pi$ then you may need to measure it in the negative direction
  - If it is $0 < \arg z \leq 2\pi$ then you will always measure in the positive direction (counter - clockwise)

Worked example

a) Find the modulus and argument of $z = 2 + 3i$

$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

Draw a sketch to help find the argument:



The argument is the counter-clockwise angle taken from the positive x-axis

$\theta = \tan^{-1}\left(\frac{3}{2}\right)$

$= 0.9827...$

$\text{Mod } z = |z| = \sqrt{13}$

$\arg z = \theta = 0.983 \text{ (3sf)}$

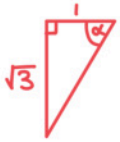
b) Find the modulus and argument of $w = -1 - \sqrt{3}i$

$$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4}$$



If the argument is measured clockwise from the positive x-axis then it will be negative.

Find this angle first and subtract from π.



$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\text{Mod}\, z = |z| = 2$$

$$\arg z = -\theta = -\frac{2\pi}{3}$$

1.8.3 Introduction to Argand Diagrams

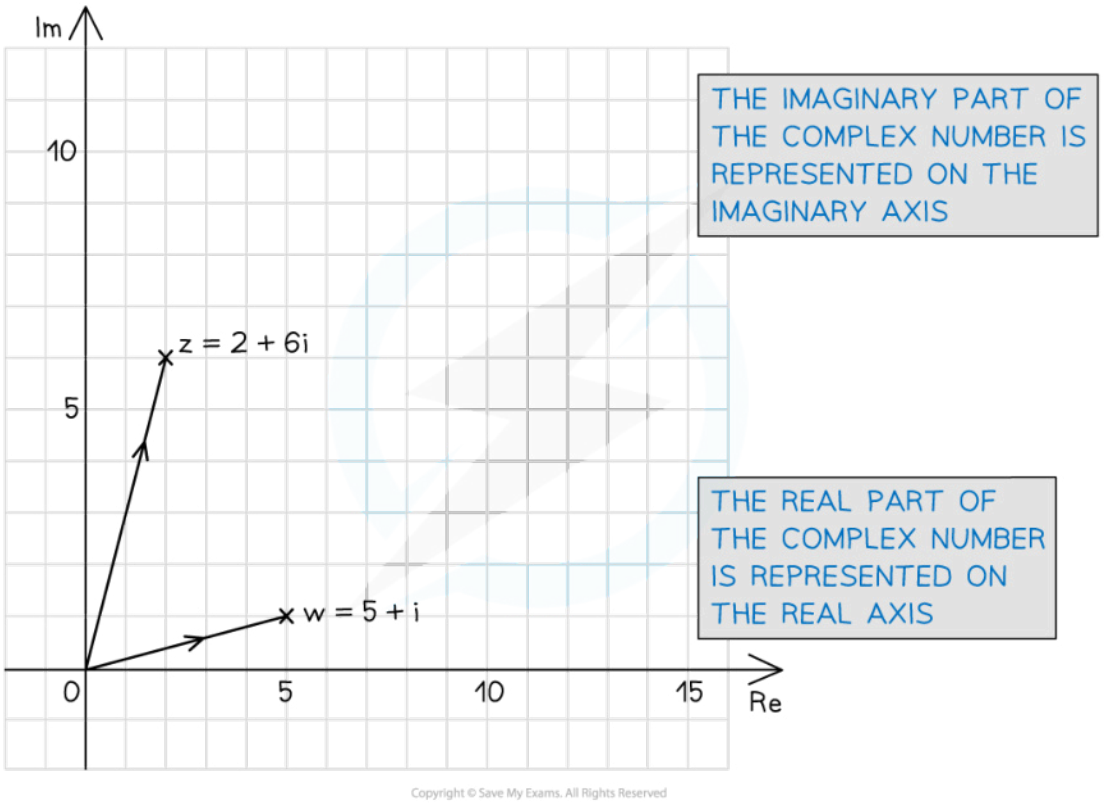
Argand Diagrams

What is the complex plane?

- The complex plane, sometimes also known as the Argand plane, is a two-dimensional plane on which complex numbers can be represented geometrically
- It is similar to a two-dimensional Cartesian coordinate grid
  - The x-axis is known as the **real** axis (Re)
  - The y-axis is known as the **imaginary** axis (Im)
- The complex plane emphasises the fact that a complex number is two dimensional
  - i.e it has two parts, a real and imaginary part
  - Whereas a real number only has one dimension represented on a number line (the x-axis only)

What is an Argand diagram?

- An Argand diagram is a geometrical representation of complex numbers on a **complex plane**
  - A complex number can be represented as either a point or a vector
- The complex number $x + yi$ is represented by the point with cartesian coordinate (x, y)
  - The **real** part is represented by the point on the **real** $(x-)$ axis
  - The **imaginary** part is represented by the point on the **imaginary** $(y-)$ axis
- Complex numbers are often represented as **vectors**
  - A line segment is drawn from the origin to the cartesian coordinate point
  - An arrow is added in the direction away from the origin
  - This allows for geometrical representations of complex numbers



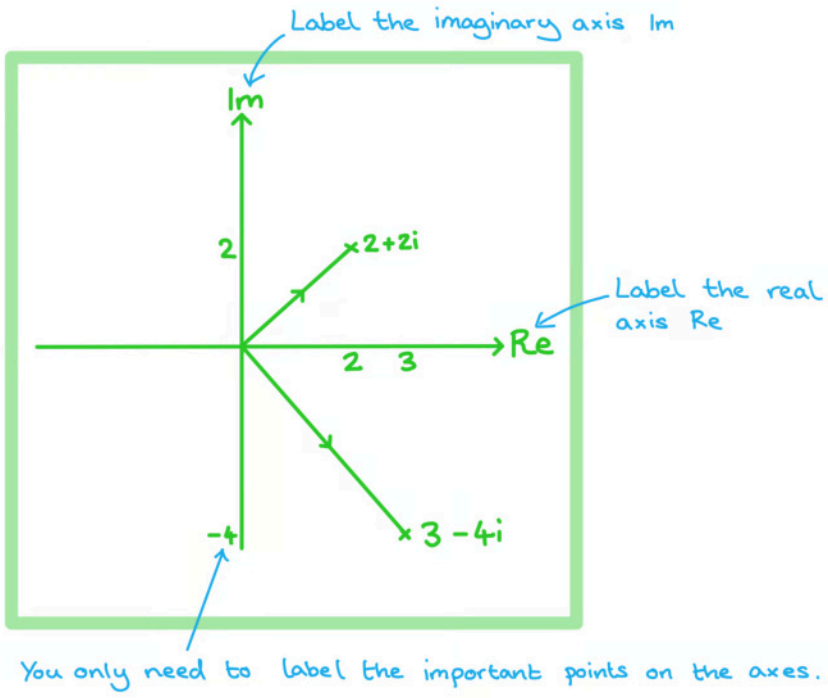
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Examiner Tip

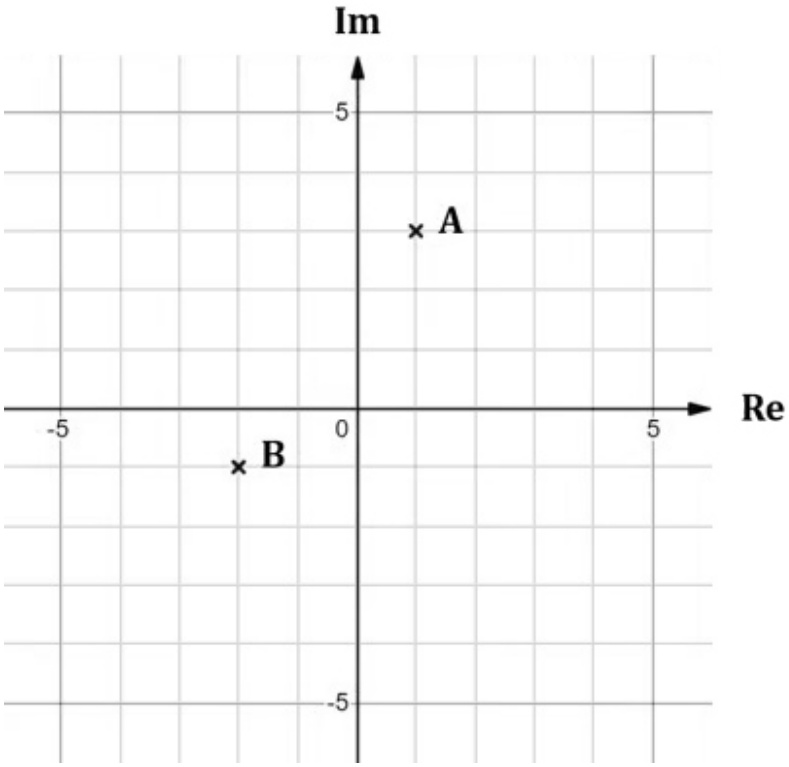
- When setting up an Argand diagram you do not need to draw fully scaled axes, you only need the essential information for the points you want to show, this will save a lot of time

Worked example

a) Plot the complex numbers $z_1 = 2 + 2i$ and $z_2 = 3 - 4i$ as points on an Argand diagram.



b) Write down the complex numbers represented by the points A and B on the Argand diagram below.



A: $1+3i$
B: $-2-i$