[Maximum mark: 4]
Consider two consecutive positive integers, k and $k+1$.
Show that the difference of their squares is equal to the sum of the two integers.

[Maximum mark: 4]
Prove that the sum of three consecutive positive integers is divisible by 3.

[Maximum mark: 4]
The product of three consecutive integers is increased by the middle integer.
Prove that the result is a perfect cube.

[Maximum mark: 6]

1.
Show that $(2n-1)^3+(2n+1)^3=16n^3+12n$ for $n\in\mathbb{Z}$. [3]
2.
Hence, or otherwise, prove that the sum of the cubes of any two consecutive odd integers is divisible by four. [3]

[Maximum mark: 6]
Using mathematical induction, prove that $1^2+2^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ for all $n\in\mathbb{Z}^+$.

[Maximum mark: 7]
Use the principle of mathematical induction to prove that

$$1\cdot 2^1+2\cdot 2^2+3\cdot 2^3+\cdots+n\cdot 2^n=2+(n-1)2^{n+1}\qquad \text{for all } n\in\mathbb{Z}^+.$$

[Maximum mark: 6]
Let $r \in \mathbb{R}, r \neq 1$. Use the method of mathematical induction to prove that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \qquad \text{for all } n \in \mathbb{Z}^+.$$

[Maximum mark: 4]
Using the method of proof by contradiction, prove that $\sqrt{3}$ is irrational.

[Maximum mark: 7]
The Fibonacci sequence is defined as follows:

$$a_0 = 0,\ a_1 = 1,\ a_2 = 1,$$

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2. \ (FS)$$

Prove by mathematical induction that $a_1^2 + a_2^2 + \cdots + a_n^2 = a_n a_{n+1}$, where $n \in \mathbb{Z}^+$.

[Maximum mark: 6]
Prove by contradiction that the equation $3x^3 - 7x^2 + 5 = 0$ has no integer roots.

[Maximum mark: 8]
Let $y = x^2e^x$, for $x \in \mathbb{R}$.

1.
Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [1]

2.
Prove by mathematical induction that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2e^x\right) = \left(n(n-1) + 2nx + x^2\right)e^x \quad \text{for all } n \in \mathbb{Z}^+, n \geq 2.$$

[7]

[Maximum mark: 9]
Let $f(x) = (x+1)e^{-2x}$, $x \in \mathbb{R}$.

1.
Find $f'(x)$. [2]

2.
Prove by induction that $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n} = \left[n(-2)^{n-1} + (-2)^n(x+1)\right]e^{-2x}$ for all $n \in \mathbb{Z}^+$. [7]

[Maximum mark: 6]
Using the principle of mathematical induction, prove that $n(n^2+5)$ is divisible by 6 for all integers $n \geq 1$.

[Maximum mark: 8]

1.
Solve the inequality $x^2 \geq 2x + 3$. [2]

2.
Use mathematical induction to prove that $2^n > n^2 - 2$ for all $n \in \mathbb{Z}^+$, $n \geq 3$. [6]

[Maximum mark: 14]

1.
Show that $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n}$, where $n \in \mathbb{Z}$, $n \geq 0$. [3]

2.
Hence show that $\frac{1}{\sqrt{2}} < 2\sqrt{2} - 2$. [2]

3.
Prove by mathematical induction that

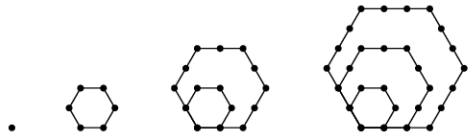
$$\sum_{r=2}^{n} \frac{1}{\sqrt{r}} < 2\sqrt{n} - 2 \qquad \text{for all } n \in \mathbb{Z}^+, n \geq 2.$$

[9]

[Maximum mark: 24]

This question asks you to investigate some properties of hexagonal numbers.

Hexagonal numbers can be represented by dots as shown below where h_n denotes the nth hexagonal number, $n \in \mathbb{N}$.



h_1 h_2 h_3 h_4

Note that 6 points are required to create the regular hexagon h_2 with side of length 1, while 15 points are required to create the next hexagon h_3 with side of length 2, and so on.

1. Write down the value of h_5. [1]

2. By examining the pattern, show that $h_{n+1} = h_n + 4n + 1$, $n \in \mathbb{N}$. [3]

3. By expressing h_n as a series, show that $h_n = 2n^2 - n$, $n \in \mathbb{N}$. [3]

4. Hence, determine whether 2016 is a hexagonal number. [3]

5. Find the least hexagonal number which is greater than $80\,000$. [5]

6. Consider the statement:
45 is the only hexagonal number which is divisible by 9.
Show that this statement is false. [2]

Matt claims that given $h_1 = 1$ and $h_{n+1} = h_n + 4n + 1$, $n \in \mathbb{N}$, then

$$h_n = 2n^2 - n, \quad n \in \mathbb{N}.$$

[7]

1. Show, by mathematical induction, that Matt's claim is true for all $n \in \mathbb{N}$.

[Maximum mark: 23]
Let $f(x) = (x-1)e^{\frac{x}{3}}$, for $x \in \mathbb{R}$.

1.
Find $f'(x)$. [2]

2.
Prove by induction that $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n} = \left(\dfrac{3n+x-1}{3^n}\right)e^{\frac{x}{3}}$ for all $n \in \mathbb{Z}^+$. [7]

3.
Find the coordinates of any local maximum and minimum points on the graph of $y = f(x)$. Justify whether such point is a maximum or a minimum. [5]

4.
Find the coordinates of any points of inflexion on the graph of $y = f(x)$. Justify whether such point is a point of inflexion. [5]

5.
Hence sketch the graph of $y = f(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [4]

[Maximum mark: 21]

1.
Use de Moivre's theorem to find the value of $\left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]^{12}$. [2]

2.
Use mathematical induction to prove that

$$(\cos\alpha - \mathrm{i}\sin\alpha)^n = \cos(n\alpha) - \mathrm{i}\sin(n\alpha) \quad \text{for all } n \in \mathbb{Z}^+.$$

[6]

Let $w = \cos\alpha + \mathrm{i}\sin\alpha$.

1.
Find an expression in terms of α for $w^n - (w^*)^n$, $n \in \mathbb{Z}^+$, where w^* is the complex conjugate of w. [2]

2.
1.
Show that $ww^* = 1$.

2.
Write down and simplify the binomial expansion of $(w - w^*)^3$ in terms of w and w^*.

3.
Hence show that $\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$. [5]

3.
Hence solve $4\sin^3\alpha + (2\cos\alpha - 3)\sin\alpha = 0$ for $0 \leq \alpha \leq \pi$. [6]

[Maximum mark: 21]

Let $f(x) = \frac{1}{\sqrt{1-x}}, x < 1$.

1.

Show that $f''(x) = \frac{3}{4}(1-x)^{-5/2}$. [3]

2.

Use mathematical induction to prove that [9]

$$f^{(n)}(x) = \left(\frac{1}{4}\right)^n \frac{(2n)!}{n!}(1-x)^{-1/2-n} \quad n \in \mathbb{Z},\ \ n \geq 2.$$

Let $g(x) = \cos(mx)$, $m \in \mathbb{Q}$.

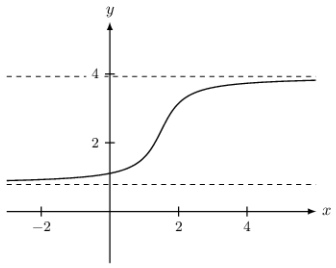
Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x < 1$.

The x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $-\frac{3}{4}$. [9]

1. Find the possible values of m.

[Maximum mark: 17]

The following diagram shows the graph of $y = \arctan(2x - 3) + \frac{3\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = \frac{\pi}{4}$ and $y = \frac{5\pi}{4}$.



1.
Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x - 3) + \frac{3\pi}{4}$ for $x \in \mathbb{R}$. [3]

2.
Show that $\arctan p - \arctan q \equiv \arctan\left(\frac{p - q}{1 + pq}\right)$. [3]

3.
Verify that $\arctan(x + 2) - \arctan(x + 1) = \arctan\left(\frac{1}{(x + 1)^2 + (x + 1) + 1}\right)$. [3]

4.
Using mathematical induction and the results from **part (b)** and **(c)**, prove that [8]

$$\sum_{r=1}^{n} \arctan\left(\frac{1}{r^2 + r + 1}\right) = \arctan(n + 1) - \frac{\pi}{4} \quad \text{for } n \in \mathbb{Z}^+.$$

Markscheme

Sorry.. please check revisiontown for markschemes. It was really hard scraping each dialog box and scaling correctly. Couldnt figure it out...