Chapter review

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1 Show that the equation $x(x-a-b)=1-ab$ always has real solutions for all values of a, $b \in \mathbb{R}$.

2 The polynomial $f(x)=ax^3+bx^2+cx+d$, a, b, c, $d \in \mathbb{Z}$, has two zeros, z and ω.

a Show that $z=(1-\mathrm{i})^3+(1+\mathrm{i})^3=-4$.

b Find $\omega=\dfrac{\mathrm{i}^{64}\times\mathrm{i}^{63}\times\ldots\times\mathrm{i}^{18}}{(1-2\mathrm{i})}$ in the form $a+b\mathrm{i}$, a, $b \in \mathbb{R}$.

c Hence find the polynomial f that has integer coefficients which are relatively prime.

3 The system of simultaneous equations

$\begin{cases} ax+y=2 \\ x+ay=-2 \end{cases}$, $a \in \mathbb{R}$, is given.

a Find the values of a for which the system has a unique solution and find the solution in terms of a.

b For which value of a does the system have:

i no solution

ii infinitely many solutions. Write the equation of the line.

4 a Prove that $\dfrac{(1+\mathrm{i})^{2019}}{(1-\mathrm{i})^{2017}}$ is an integer and find its value.

b What must $n \in \mathbb{Z}$ satisfy so that $\dfrac{(1+\mathrm{i})^{n+2}}{(1-\mathrm{i})^{n}}$ is an imaginary number?

5 Use your GDC to solve the inequality $x^5+3x^4-2x^2+2x+4 \geq x^4+3x^3+2x^2+3$.

6 Solve the simultaneous equations.

$\begin{cases} 3x+z=2y+1 \\ 6x+8y=3z+6 \\ 4y-7z=12x+4 \end{cases}$

7 The polynomial f is such that $(f(x))^2=9x^4+12x^3-26x^2-20x+25$.

a What is the degree of the polynomial f?

b Given that α and β are the zeros of the polynomial f, find $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$.

8 Find the complex numbers z and ω such that $\begin{cases} z^*-\mathrm{i}\omega=3 \\ \omega^*+\mathrm{i}z=6-\mathrm{i} \end{cases}$

9 Given that $z^2-z+1=0$, show that $z^{2019}=-1$.

Exam-style questions

10 **P1: a** Show that the solutions to the equation $x^2-6x-43=0$ may be written in the form $x=p\pm q\sqrt{13}$, where p and q are positive integers. (4 marks)

b Hence, or otherwise, solve the inequality $x^2-6x-43 \leq 0$. (2 marks)

11 **P1: a** Solve the equation $8x^2+6x-5=0$ by factorization. (4 marks)

b Determine the range of values of k for which $8x^2+6x-5=k$ has no real solutions. (3 marks)

12 **P1:** Find the range of values of k for which the equation $3kx^2+k\sqrt{3}x+3=0$ has two real roots. (6 marks)

13 **P1: a** Solve the inequality $(x+4)(3-x)>0$. (3 marks)

b Solve the inequality $2x^2-11x+9<0$. (4 marks)

c Hence, find the range of values of x which satisfy both the inequalities $(x+4)(3-x)>0$ and $2x^2-11x+9<0$. (1 mark)



14 **P1:** Find the possible values of $\omega \in \mathbb{C}$ which satisfy $\omega^2 = 77 - 36i$. (9 marks)

15 **P1:** The polynomial $2x^3 + ax^2 - 10x + b$ has a factor of $(x - 1)$, and has a remainder of 15 when divided by $(x + 2)$. Find the values of a and b ($a, b \in \mathbb{R}$). (7 marks)

16 **P1:** $f(z) = z^4 - 8z^3 + 48z^2 - 176z + 260$, $z \in \mathbb{C}$

Given $(3 - i)$ is a zero of f, find the remaining zeros. (9 marks)

17 **P1:** Find the sum and product of the roots of each function.

a $f(x) = 5x4 - 4x^3 + 3x^2 + 2x - 1$ (4 marks)

b $g(x) = 5x^5 - 4x^4 + 3x^3 + 2x^2 + -7x + 10$ (4 marks)

18 **P1:** Prove that $\omega z^* - z\omega^*$ is purely imaginary for all $\omega, z \in \mathbb{C}$. (5 marks)

19 **P2:** Find the equation of a quadratic function $f(x) = ax^2 + bx + c$ passing through the points $(-1, -5)$, $(3, -1)$, and $(10, -71)$. (7 marks)

Making a Mandelbrot

Approaches to learning: Critical thinking, Communication

Exploration criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB topic: Complex numbers



Fractals

You may have heard about fractals.

The image above is from the Mandelbrot set, one of the most famous examples of a fractal.

This is not only a beautiful image in its own right. The Mandelbrot set as a whole is an object of great interest to mathematicians. However, as yet, no practical applications have been found.

This image appears to be very complicated, but is in fact created using a remarkably simple rule.

Exploring an iterative equation

Consider this iterative equation:

$z_{n+1} = (z_1)^2 + c$

Consider a value of $c = 0.5$, so you have $z_{n+1} = (z_1)^2 + 0.5$.

Given that $z_1 = 0$, find the value of z_2.

Now find the value of z_3.

Repeat for a few more iterations.

What do you think is happening to the values found in this calculation?

A different iterative equation

Now consider a value of $c = -0.5$, so you have $z_{n+1} = (z_1)^2 - 0.5$.

Again start with $z_1 = 0$.

What happens this time?

The Mandelbrot set

The Mandelbrot set consists of all those values of c for which the sequence starting at $z_1 = 0$ does **not** escape to infinity (those values of c where the calculations zoom off to infinity).

That is only part of the story however.

The value of c in the function does not need to be real.

It could be complex of the form $a + b$i.

If you pick a complex number for c, is it in the Mandelbrot set or not?

Consider starting with a value of $z_1 = 0 + 0$i, rather than just "0".

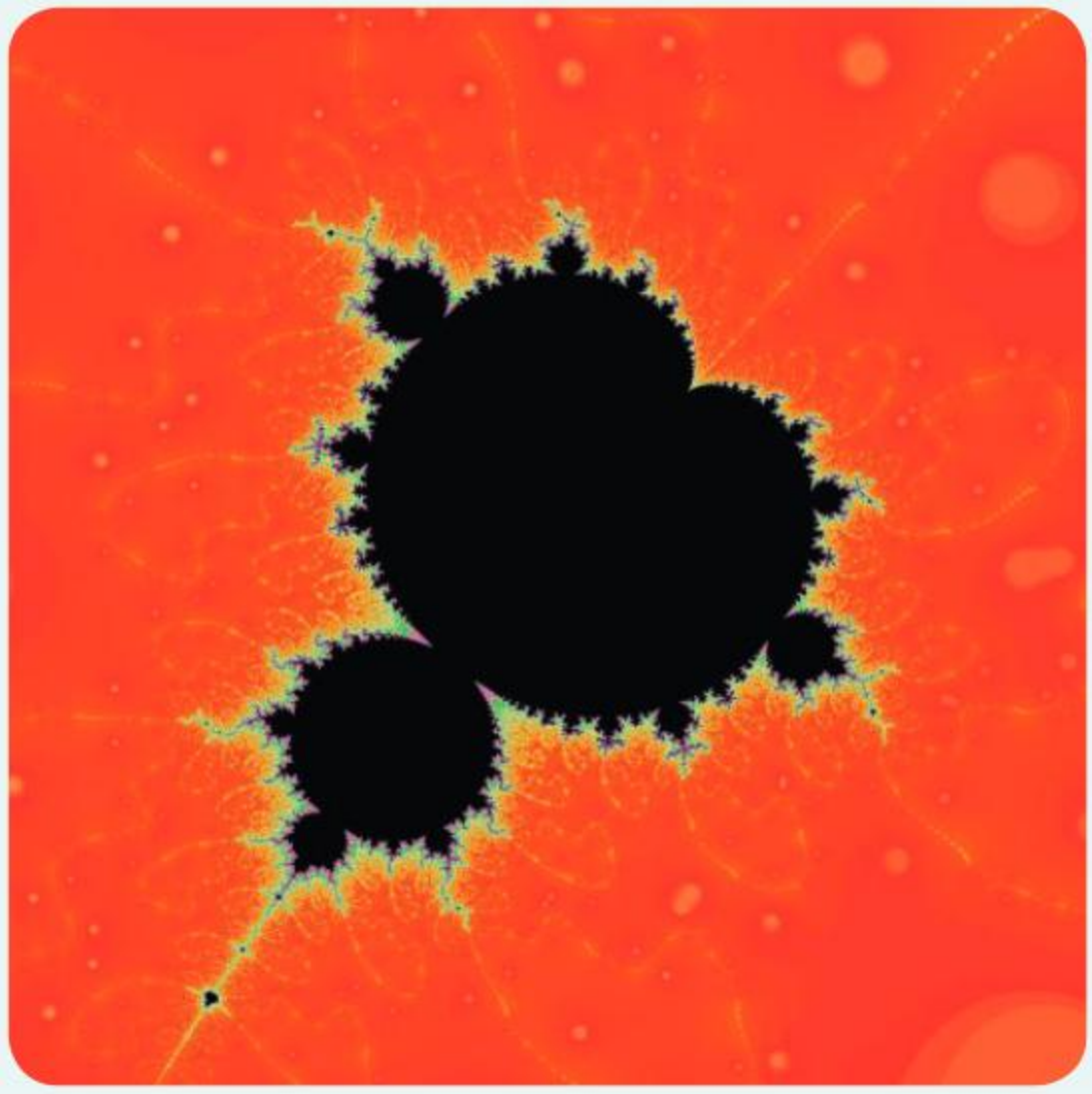
Consider, for example, a value of c of $1 + $i, so you have $z_{n+1} = (z_1)^2 + (1 + i)$.

$z_1 = 0 + 0$i

Find z_2.

Repeat to find z_3, z_4, etc.

Does this diverge or converge?
(Does it zoom off to infinity or not?)



HINT

One way to check is to plot the different values of z_1, z_2, z_3, z_4, etc, on an Argand diagram and to see whether the points remain within a boundary square. Or you could calculate the modulus of each value and see whether this is increasing.

Repeat for a few more values of c.

Try, for example:

a $c = 0.2 - 0.7$i **b** $c = -0.25 + 0.5$i

What happens if $c =$ i?

Try some values of your own.

You may find a GDC will help with these calculations as it can be used for complex numbers. When the number gets very big, the calculator will not be able to give a value. You may also notice that some numbers get "big" a lot quicker than others.

It is clearly a time-consuming process to try all values!

There are programmes that can be used to calculate the output after several iterations when you input a number for c. This will indicate which values of c belong to the Mandelbrot set and which don't.

The Mandelbrot set diagram is created by colouring points on an Argand diagram of all those values of c that escape to infinity in one colour (say black) and all those that remain bounded in another colour (say red).

Extension

- Try to construct your own spreadsheet or write a code that could do this calculation a number of times for different chosen values of c.
- Explore what happens as you zoom in to the edges of the Mandelbrot set.
- What is the relationship between Julia sets and the Mandelbrot set?
- What is the connection to Chaos Theory?
- What are Multibrot sets?
- How could you find the area or perimeter of the Mandelbrot set?