3 Expanding the number system: complex numbers

So far you have only worked with real numbers. In this chapter, you will begin to work with complex numbers: a new type of number that complies with all the properties of real numbers, but expands into another dimension. When complex numbers were first discovered they were called imaginary numbers since mathematicians could not find a real application for them. At a later stage, these numbers were put to use in many practical areas, and their name was changed to complex numbers. You will discover that many equations that we could not find a solution to within the set of real numbers actually have complex solutions. In this chapter you will also learn about one of the most important theorems in mathematics: the fundamental theorem of algebra.

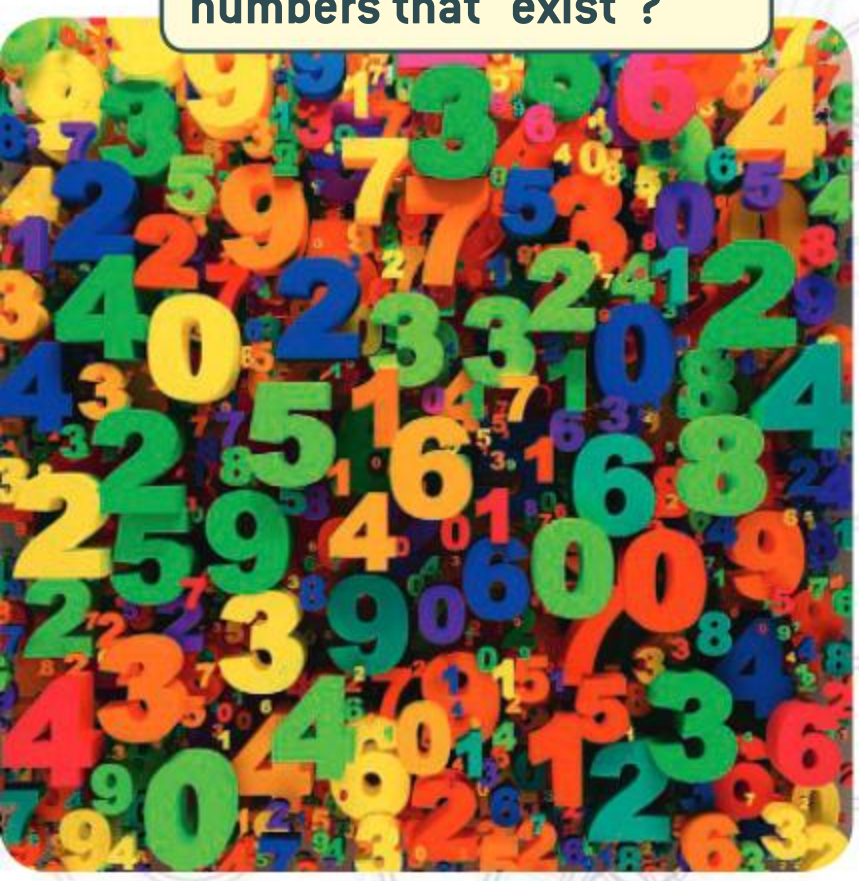
Concepts

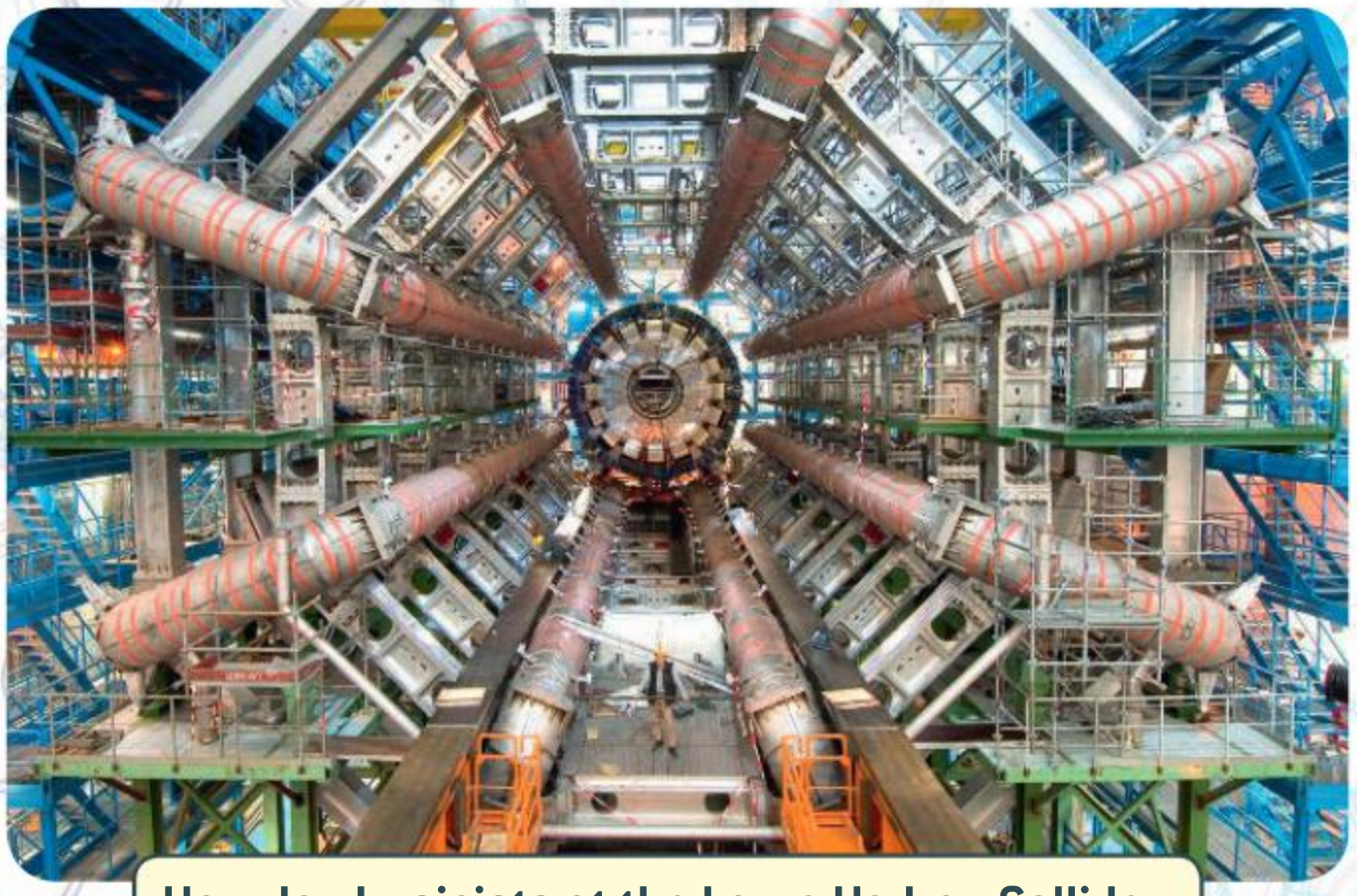
- Systems
- Patterns

Microconcepts

- Quadratic function and graph
- Quadratic equations and inequalities
- Discriminant
- Complex numbers
- Modulus of a complex number
- Operations with complex numbers
- Powers and roots of complex numbers
- Polynomial functions and their graphs
- Operations on polynomials
- Linear combination of two polynomials
- Factor and remainder theorem
- The fundamental theorem of algebra
- Polynomial equations
- Sum and product of the roots of polynomial equations
- Polynomial inequalities
- Simultaneous equations

Are real numbers the only numbers that "exist"?





How do physicists at the Large Hadron Collider at CERN model the smashing of particles?

Your city council has asked you to prepare a plan of a roller coaster for a new amusement park. There are some constraints regarding the height of the roller coaster.

The designer would like to model the track using functions. What kind of features are impossible to have, given this restriction?

What kinds of functions are suitable for modelling the roller coaster and why?

Is it possible to model the entire track by using one function only?

If it is not possible, what conditions must the piecewise functions have at their joining points?

How many points are necessary to determine the equation of a curved part of the track?



Developing inquiry skills

Write down any similar inquiry questions you might ask to model another path—for example, the path of a character in a 2D computer game. What different questions might you need to ask?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

Click here for help with this skills check

You should know how to:

1 Solve simple quadratic equations, eg

a $x^2 = 225 \Rightarrow x = \pm\sqrt{225} = \pm 15$

b $3x^2 - 72 = 0 \Rightarrow x^2 = 24$

$\Rightarrow x = \pm\sqrt{24} = \pm 2\sqrt{6}$

2 Use a GDC to graph linear and quadratic functions, and to solve equations, eg

Given the functions $f(x) = x^2 - 2x + 2$ and $g(x) = -2x + 3$ find the values of x such that $f(x) = g(x)$.

Graph both functions and find the points of intersection.

Skills check

1 Solve the following quadratic equations.

a $x^2 = 169$

b $7x^2 = 196$

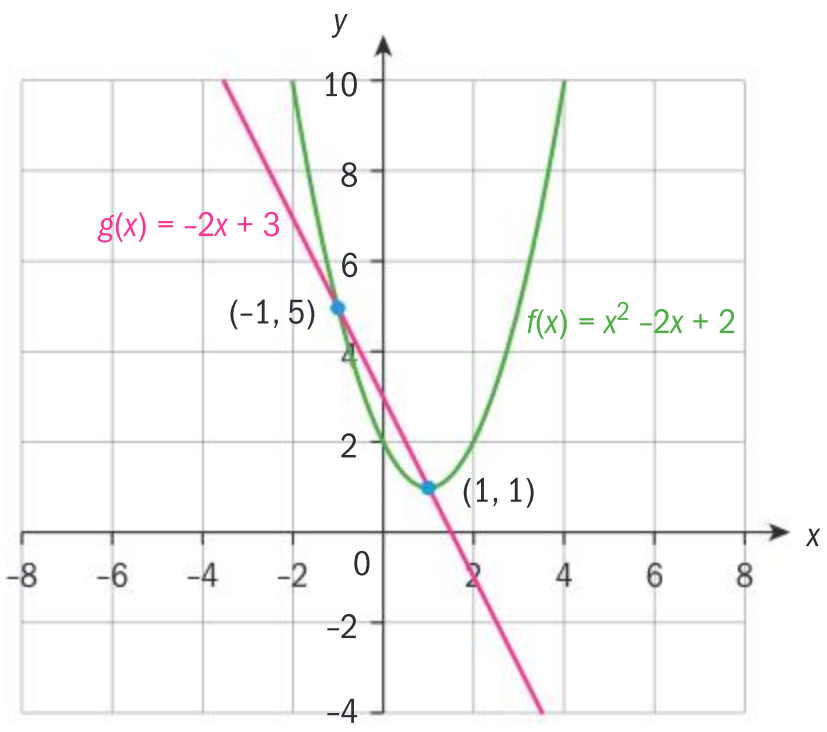
c $11x^2 + 13 = 1102$

2 For the functions f and g, find the values of x such that $f(x) = g(x)$.

a $f(x) = 2x^2 - x - 1$ and $g(x) = -3x + 3$

b $f(x) = 4x^2 - x - 1$ and $g(x) = x^2 + x$

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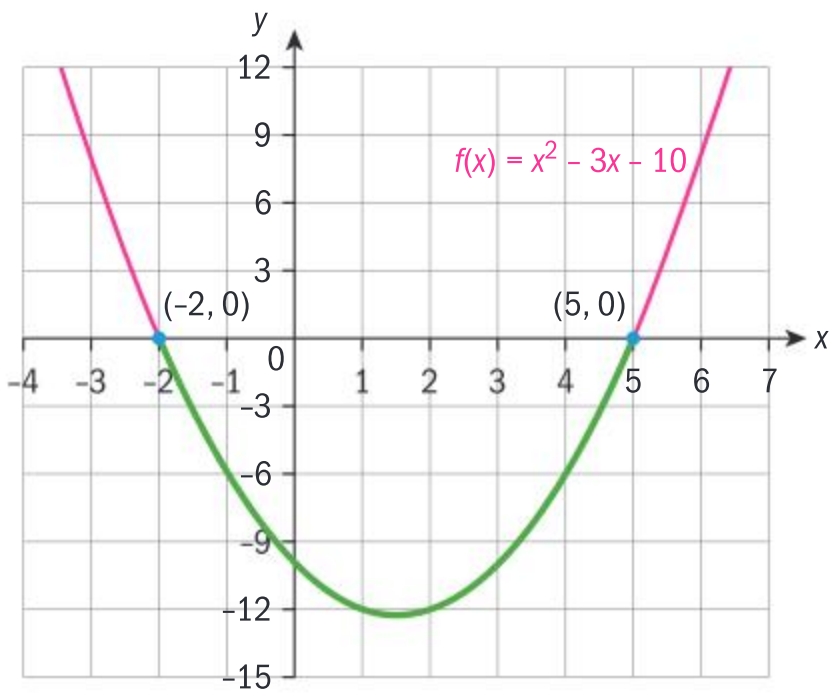


$x = -1$ or $x = 1$

3 Use a GDC to solve linear or quadratic inequalities, eg

Find the values of x such that $x^2 - 3x \leq 10$.

Rewrite the inequality as a quadratic function and find all the values of x for which the graph is on or below the x-axis.



$x \in [-2, 5]$.

4 Solve simultaneous equations, eg

$$\begin{cases} 3x + 4y = 5 \\ 2x + 3y = -1 \end{cases}$$

Use one of the methods to solve, eg the method of elimination.

$$\begin{cases} 3x + 4y = 5 \\ 2x + 3y = -1 \end{cases} \Rightarrow \begin{cases} 6x + 8y = 10 \\ -6x - 9y = 3 \end{cases}$$

$-y = 13 \Rightarrow y = -13 \Rightarrow 2x + 3 \times (-13) = -1$

$\Rightarrow 2x = 38 \Rightarrow x = 19 \Rightarrow (x, y) = (19, -13)$

3 Find the values of x such that

a $2x - 1 > 3x + 2$

b $x^2 + 2x \leq 8$

c $3x^2 - 2x \leq 2 + 5x - x^2$

4 Solve the following simultaneous equations by a method of your choice.

a $\begin{cases} 7x - 4y = 1 \\ 5x - 3y = 2 \end{cases}$

b $\begin{cases} 2x + 3y = 1 \\ 6x = 3 - 9y \end{cases}$

c $\begin{cases} 2x - y = 4 \\ 2x + 3y = 1 \end{cases}$

d $\begin{cases} x = 5y + 1 \\ 2x - 3 = 10y \end{cases}$



3.1 Quadratic equations and inequalities

International-mindedness

The Babylonians (2000–1600 BC) used quadratics to find the area of fields for agriculture and taxation.

Solving quadratic equations by factorization

When a quadratic equation can be expressed as the product of two linear factors, you can factorize the equation and solve it.

Example 1

Solve the following quadratic equations by factorization.

a $x^2 + 3x - 40 = 0$ **b** $2x^2 - 7x - 4 = 0$

a $x^2 + 3x - 40 = 0 \Rightarrow x^2 + 8x - 5x - 40 = 0$ $\Rightarrow x(x + 8) - 5(x + 8) = 0$ $\Rightarrow (x + 8)(x - 5) = 0$	Split the middle term, so that you can factorize a common factor from the first two terms, and a common factor from the last two terms. Use distribution to factorize the expression.
$x + 8 = 0$ or $x - 5 = 0 \Rightarrow x = -8$ or $x = 5$	Use the zero product theorem to find the solutions.
b $2x^2 - 7x - 4 = 0 \Rightarrow 2x^2 - 8x + x - 4 = 0$	Again, split the middle term and factorize in pairs.
$2x(x - 4) + (x - 4) = 0 \Rightarrow (2x + 1)(x - 4) = 0$	Use distribution to factorize the expression.
$2x + 1 = 0$ or $x - 4 = 0 \Rightarrow x = -\frac{1}{2}$ or $x = 4$	Use the zero product theorem to find the solutions.

Exercise 3A

Solve the following equations by factorization.

1 $x^2 + 8x + 15 = 0$

2 $x^2 + 5x - 14 = 0$

3 $3x^2 - 7x + 2 = 0$

4 $4x^2 - 20x + 25 = 0$

5 $5x^2 - 4x - 12 = 0$

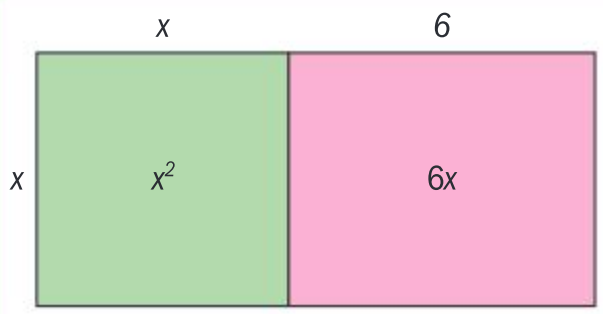
Solving equations by completing the square

Most quadratic equations that you will come across do not involve a perfect square. However, they can easily be transformed into an expression which *does* involve a perfect square by a process called **completing the square**.

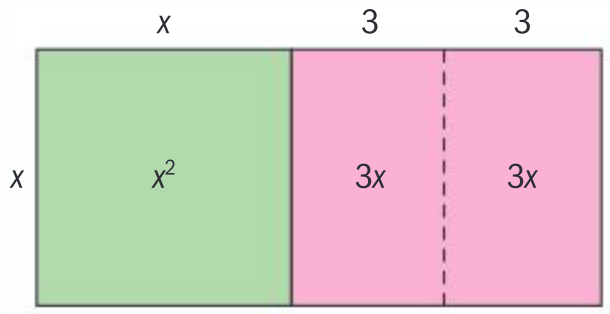
You already know how to solve a quadratic equation by factorization. However, it is not always possible to factorize a quadratic equation. Completing the square is another method that you can use to solve a quadratic equation. You can use this method whenever the equation has real solutions.

First you will find the value of c that can be added to $x^2 + bx$ to form a perfect square.

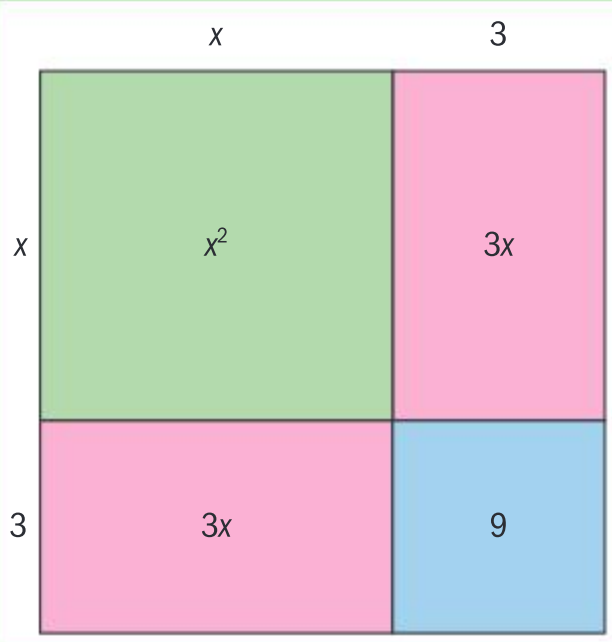
The following diagrams will show you how to do this.



The area of the region shaded in green is x^2 and the area of the region shaded in pink is $6x$. Thus, the area of the whole rectangle is $x^2 + 6x$.



Divide the pink region into two equal parts.



Rearrange the parts.

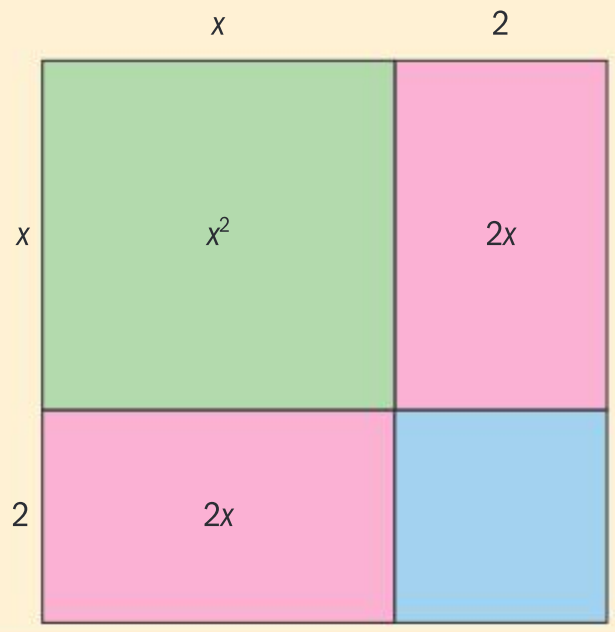
The large square formed has area $(x + 3)^2$

So the expression $x^2 + 6x$ becomes a perfect square when you add 9.

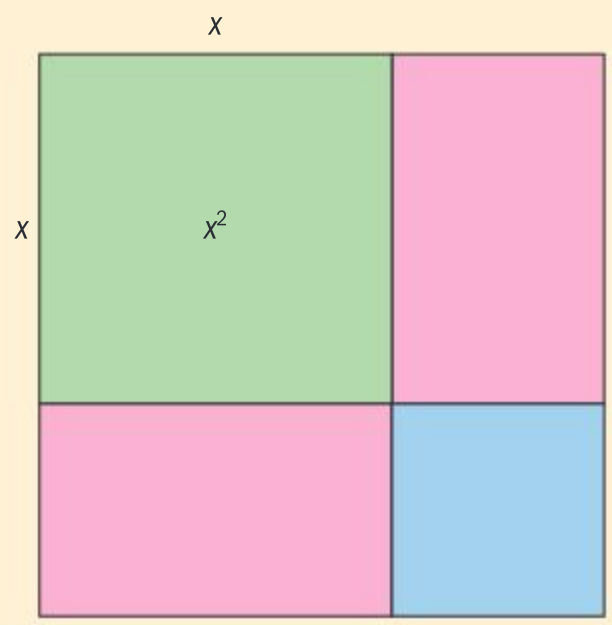
Investigation 1

For each of the following diagrams, find the value of c which makes the expression into a perfect square, and then write the expression as a perfect square.

1 $x^2 + 4x + c$

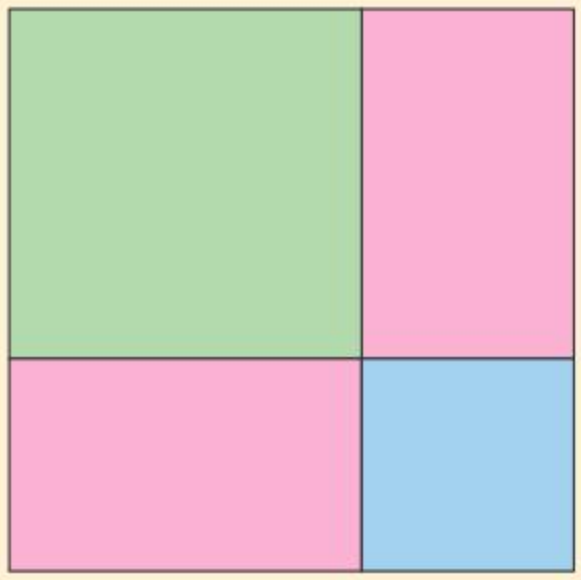


2 $x^2 + 10x + c$

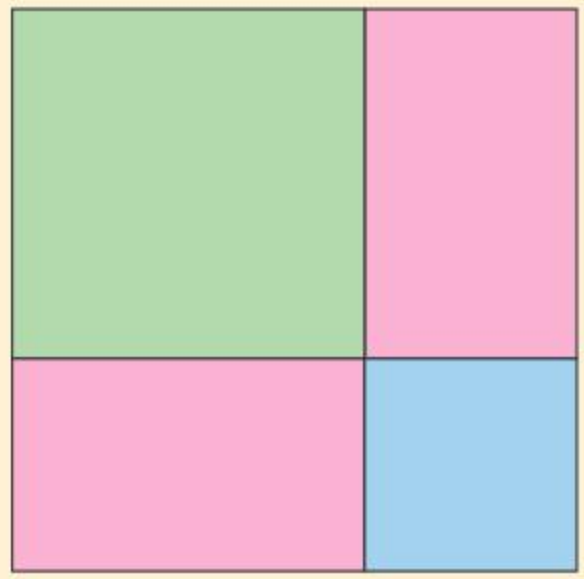




3 $x^2 + 3x + c$



4 $x^2 + bx + c$



5 Fill in the blank: To complete the square for $x^2 + bx$, add _____.

6 **Factual** What are the steps of the completing the square method?

7 **Conceptual** How does the area model help you understand the process of completing the square and why is the process called completing the square?

Use your findings from the investigation to algebraically solve the following examples.

Example 2

For each quadratic expression, find the value of k which makes the expression a perfect square. Hence, write each in a factorized form.

a $x^2 + 8x + k$ **b** $x^2 - 7x + k$ **c** $9x^2 + 3x + k$

a $x^2 + 8x + k = x^2 + \overbrace{2 \times x \times 4}^{8x} + \overbrace{4^2}^{k} = (x+4)^2$

$\Rightarrow k = 4^2 = 16$

b $x^2 - 7x + k = x^2 \overbrace{-2 \times x \times \frac{7}{2}}^{-7x} + \overbrace{\left(\frac{7}{2}\right)^2}^{k}$

$= \left(x - \frac{7}{2}\right)^2 \Rightarrow k = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$

c $9x^2 + 3x + k = (3x)^2 + \overbrace{2 \times (3x) \times \frac{1}{2}}^{3x} + \overbrace{\left(\frac{1}{2}\right)^2}^{k}$

$= \left(3x + \frac{1}{2}\right)^2 \Rightarrow k = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

Use the binomial expansion and express the term in x as twice the product of x with a constant term. Use this constant term to find the value of k.

Once you have transformed a quadratic equation to a form which involves a perfect square, you can easily solve the equation.

Example 3

Solve the following equations by completing the square. Give your answers in exact form.

a $x^2 + 5x - 24 = 0$ **b** $2x^2 - 7x + 1 = 0$

a $x^2 + 5x - 24 = 0 \Rightarrow x^2 + 5x = 24$

$\Rightarrow x^2 + 2 \times x \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 = 24 + \left(\frac{5}{2}\right)^2$

$\Rightarrow \left(x + \frac{5}{2}\right)^2 = \frac{121}{4} \Rightarrow x + \frac{5}{2} = \pm\sqrt{\frac{121}{4}}$

$\Rightarrow x = -\frac{5}{2} \pm \frac{11}{2} \Rightarrow x = -8$ or $x = 3$

Move the term independent of x to the right-hand side.

Complete the square on the left-hand side, and add the same constant term to the right-hand side so that you don't change the equation.

Factorize the complete square and solve the equation.

b $2x^2 - 7x + 1 = 0 \Rightarrow x^2 - \frac{7}{2}x + \frac{1}{2} = 0$

$\Rightarrow x^2 - 2 \times x \times \frac{7}{4} + \left(\frac{7}{4}\right)^2 = -\frac{1}{2} + \left(\frac{7}{4}\right)^2$

$\Rightarrow \left(x - \frac{7}{4}\right)^2 = \frac{41}{16} \Rightarrow x - \frac{7}{4} = \pm\sqrt{\frac{41}{16}}$

$\Rightarrow x = \frac{7 \pm \sqrt{41}}{4}$

Since the coefficient of x^2 is not 1, divide the whole equation by 2.

Move the term independent of x to the right-hand side.

Complete the square on the left-hand side, and add the same constant term to the right-hand side.

Factorize the expression and solve the equation.

There is no need to separate the solutions since they are written in the surd conjugate form.

In Example **3b**, you left your answer in surd form. However, when solving real-life problems, it is much more practical to give your answers in decimal form to an appropriate degree of accuracy. These values can then be used in making measurements, or whatever the context of the problem is.

Exercise 3B

1 Solve the following equations by completing the square. Give your answers in exact form.

a $x^2 + 6x - 7 = 0$

b $x^2 - 7x - 30 = 0$

c $x^2 - x - 1 = 0$

d $3x^2 - 7x + 2 = 0$

e $4x^2 + 12x + 5 = 0$

f $5x^2 - 10x + 2 = 0$



2 Solve the following equations by completing the square. Give your answers correct to 3 significant figures.

a $x^2 + 2x - 1 = 0$

b $(p-4)x + (p+2)y = -3$

c $2x^2 - x - 3 = 0$

d $3x^2 + 9x + 5 = 0$

The quadratic formula

Notice that you can generally solve a quadratic equation by factorization when its roots are integers, but you can *always* solve a quadratic equation by completing the square – even when its roots are surds. However, the disadvantage is that completing the square can be long and repetitive. It is, therefore, helpful to have a quicker way to find the solutions to any quadratic equation with real roots.

Investigation 2

In this investigation, you will start with the general form of a quadratic equation, $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$, and use completing the square to find its solutions in terms of a, b and c.

Copy the following table and fill in the missing steps or explanations.

Mathematical working	Explanation
$ax^2 + bx + c = 0$	
1	**1** Multiply both sides of the equation by a.
2	**2** Subtract ac from both sides of the equation.
3 $a^2x^2 + 2 \times ax \times \frac{b}{2} + \left(\frac{b}{2}\right)^2 = -ac + \left(\frac{b}{2}\right)^2$	**3**
4 $\left(ax + \frac{b}{2}\right)^2 = \frac{b^2}{4} - ac$	**4**
5 $\left(ax + \frac{b}{2}\right)^2 = \frac{b^2 - 4ac}{4}$	**5**
6	**6** Take the square root of both sides.
7 $ax = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4ac}}{2}$	**7**
8	**8** Divide by a to find x.

Factual What do a, b and c stand for in your formula in line 8?

Conceptual How does the quadratic formula generalize the solutions to quadratic equations?

HINT

When solving quadratic equations the solutions are called roots since in the formula you use a square root to find them. When using quadratic functions, the points of intersection with the x-axis have x values that are obtained by solving the corresponding quadratic equation. In this case you call them zeros because the value of the function for those x values is 0.

HINT

You could take the same approach that we used in Example **3b**, by dividing the whole equation by a, since $a \neq 0$. However, this approach will guide you through using a slightly different method which will give the same result.

TOK

How can you deal with the ethical dilemma of using mathematics to cause harm, such as plotting the course of a missile?

The quadratic formula

For a quadratic equation in the form $ax^2 + bx + c = 0,\ a, b, c \in \mathbb{R},\ a \neq 0$, the solutions or roots are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 4

Use the quadratic formula to solve the following equations. Leave your answers in exact form.

a $x^2 + 11x + 24 = 0$ **b** $3x^2 + 8x + 4 = 0$ **c** $5x^2 - 2x - 1 = 0$ **d** $2x^2 + x + 1 = 0$

a $x^2 + 11x + 24 = 0 \Rightarrow a = 1,\ b = 11,\ c = 24$	Identify the coefficients a, b and c and apply the formula.
$\Rightarrow x = \dfrac{-11 \pm \sqrt{11^2 - 4 \times 1 \times 24}}{2 \times 1} = \dfrac{-11 \pm \sqrt{121 - 96}}{2}$	
$= \dfrac{-11 \pm 5}{2} \Rightarrow x = -8$ or $x = -3$	Separate the solutions.
b $3x^2 + 8x + 4 = 0 \Rightarrow a = 3,\ b = 8,\ c = 4$	Identify the coefficients a, b and c and apply the formula.
$\Rightarrow x = \dfrac{-8 \pm \sqrt{8^2 - 4 \times 3 \times 4}}{2 \times 3} = \dfrac{-8 \pm \sqrt{64 - 48}}{6}$	
$= \dfrac{-8 \pm 4}{6} \Rightarrow x = \dfrac{-12}{6} = -2$ or $x = \dfrac{-4}{6} = -\dfrac{2}{3}$	Separate the solutions.
c $5x^2 - 2x - 1 = 0 \Rightarrow a = 5,\ b = -2,\ c = -1$	Identify the coefficients a, b and c and apply the formula.
$\Rightarrow x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 5 \times (-1)}}{2 \times 5} = \dfrac{2 \pm \sqrt{4 + 20}}{10}$	
$= \dfrac{2 \pm 2\sqrt{6}}{10} \Rightarrow x = \dfrac{1 - \sqrt{6}}{5}$ or $x = \dfrac{1 + \sqrt{6}}{5}$	Separate the solutions.
d $2x^2 + x + 1 = 0 \Rightarrow a = 2,\ b = 1,\ c = 1$	Identify the coefficients a, b and c and apply the formula.
$\Rightarrow x = \dfrac{-1 \pm \sqrt{1^2 - 4 \times 2 \times 1}}{2 \times 2} = \dfrac{-1 \pm \sqrt{1 - 8}}{4}$	
$= \dfrac{-1 \pm \sqrt{-7}}{4} \Rightarrow x \notin \mathbb{R}$	Since the expression under the square root is negative, there is no real solution.

When either of the coefficients b and/or c are equal to zero, we have special cases that we solve by using simpler methods.



Case 1

$b = 0,\ c \neq 0$

$ax^2 + c = 0 \Rightarrow ax^2 = -c \Rightarrow x^2 = -\frac{c}{a} \Rightarrow x = \pm\sqrt{-\frac{c}{a}}$, where

$x \in \mathbb{R}$ if $-\frac{c}{a} \geq 0$ and $x \notin \mathbb{R}$ if $-\frac{c}{a} < 0$.

Case 2

$b \neq 0,\ c = 0$

$ax^2 + bx = 0 \Rightarrow x(ax + b) = 0 \Rightarrow x = 0$ or $ax + b = 0 \Rightarrow x = -\frac{b}{a}$, so you have two real solutions, one of which is always 0.

If one or more of the coefficients in a quadratic equation is an unknown constant, you can use the quadratic formula to find the solutions in terms of that constant.

Example 5

Solve the quadratic equation $px^2 + 3 = 3px + x$, $p \neq 0$, by using the quadratic formula. Express x in terms of the real parameter p.

$px^2 + 3 = 3px + x \Rightarrow px^2 - (3p + 1)x + 3 = 0$	Rewrite the equation in the general quadratic form.
$\Rightarrow a = p,\ b = -(3p + 1),\ c = 3$	Identify the coefficients.
$\Rightarrow x = \frac{-\left(-(3p+1)\right) \pm \sqrt{\left(-(3p+1)\right)^2 - 4 \times p \times 3}}{2 \times p}$	Apply the quadratic formula.
$= \frac{3p+1 \pm \sqrt{9p^2 + 6p + 1 - 12p}}{2p}$	Simplify.
$= \frac{3p+1 \pm \sqrt{9p^2 - 6p + 1}}{2p}$	
$= \frac{3p+1 \pm \sqrt{(3p-1)^2}}{2p}$	Factorize the expression under the square root.
$= \frac{3p+1 \pm (3p-1)}{2p} \Rightarrow x = \frac{3p+1+3p-1}{2p} = \frac{6p}{2p} = 3$	Simplify again.
or $x = \frac{3p+1-3p+1}{2p} = \frac{2}{2p} = \frac{1}{p}$	Write both solutions separately.

Exercise 3C

1 Solve the following equations by using the quadratic formula. Give your answers in the exact form.

a $x^2 + 9x + 18 = 0$ **b** $x^2 - x - 30 = 0$

c $x^2 - x - 1 = 0$ **d** $2x^2 - 3x = 2$

e $\sqrt{2}x^2 + \sqrt{3} = 11x$

2 Solve for x the following quadratic equations by using the quadratic formula. Give your answers in terms of the real parameter.

a $x^2 + 3x = ax + 3a$ **b** $2x^2 - b = x - 2bx$

c $x^2 + kx = 2k^2$ **d** $p^2x^2 + 3px = px + 3$

Discriminant of a quadratic equation

Investigation 3

1 Use the quadratic formula to solve the following equations.

a $x^2 - 3x - 10 = 0$ **b** $3x^2 - 2x + 5 = 0$

c $x^2 - 12x + 36 = 0$ **d** $4x^2 - 20x + 25 = 0$

e $6x^2 - 13x + 6 = 0$ **f** $x^2 - 5x + 7 = 0$

2 **Factual** How many real solutions can different quadratic equations have?

3 **Conceptual** By considering the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, explain why quadratic equations have different numbers of solutions.

The expression $b^2 - 4ac$ is called the **discriminant** of a quadratic.

4 Sketch graphs of $y = f(x)$ for each equation given in question **1.** On your sketch, label any zeros of each function.

5 **Factual** What do you notice about the number of zeros of each graph, and the number of solutions to the quadratic equation?

6 **Factual** What does the graph of a quadratic equation tell you about the value of the discriminant?

7 **Conceptual** What can the discriminant be used for?

Given a quadratic equation of the form $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$, the **discriminant** is the expression in the formula that is under the square root and is denoted by the Greek letter Δ, $\Delta = b^2 - 4ac$.

As you found in the investigation, the sign of the discriminant determines the number of roots of the quadratic.

International-mindedness

Frenchman Nicole Oresme was one of the first mathematicians to consider the concept of functions in the 14th century, the term "function" was introduced by the German mathematician Gottfried Wilhelm Leibniz in the 17th century and the notation was coined by Swiss Leonard Euler in the 18th century.



Case 1: $\Delta > 0$

If the discriminant is positive, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and there are **two distinct real roots.**

Case 2: $\Delta = 0$

If the discriminant is equal to zero, then $x = \dfrac{-b}{2a}$. This is regarded as **one repeated real root.**

Case 3: $\Delta < 0$

If the discriminant is less than zero, then $\sqrt{b^2 - 4ac}$ is not real. In this case, there is **no real solution**.

Example 6

Without solving, determine the nature of the roots of each equation.

a $5x^2 + 7x - 1 = 0$ **b** $25x^2 + 49 = 70x$ **c** $2x^2 + \dfrac{7}{4}x + \dfrac{1}{2} = 0$

a $5x^2 + 7x - 1 = 0 \Rightarrow a = 5,\ b = 7,\ c = -1$	Identify the coefficients.
$\Delta = 7^2 - 4 \times 5 \times (-1) = 69 > 0$	Calculate the discriminant.
There are two distinct real roots.	Since the discriminant is greater than 0 there are two distinct real roots.
b $25x^2 + 49 = 70x \Rightarrow 25x^2 - 70x + 49 = 0$	Write in the form $ax^2 + bx + c = 0$.
$\Rightarrow a = 25,\ b = -70,\ c = 49$	Identify the coefficients.
$\Delta = (-70)^2 - 4 \times 25 \times 49 = 4900 - 4900 = 0$	Calculate the discriminant.
There is one repeated real root.	Since the discriminant is equal to 0 there is one repeated real root.
c $2x^2 + \dfrac{7}{4}x + \dfrac{1}{2} = 0 \Rightarrow a = 2,\ b = \dfrac{7}{4},\ c = \dfrac{1}{2}$	Identify the coefficients.
$\Delta = \left(\dfrac{7}{4}\right)^2 - 4 \times 2 \times \dfrac{1}{2} = \dfrac{49}{16} - 4 = -\dfrac{15}{16} < 0$	Calculate the discriminant.
There are no real roots.	Since the discriminant is less than 0 there are no real roots.

You might also be asked to determine the value of a real parameter so that the equation has different types of roots.

Example 7

Find the value(s) of the real parameter p so that

a $2x^2 - 3x + p = 0$ has two real roots **b** $px^2 + p = 13x$ has one real repeated root

c $(p + 2)x^2 + 2px = 1 - p$ has no real roots.

Continued on next page

a $2x^2 - 3x + p = 0 \Rightarrow \Delta = (-3)^2 - 4 \times 2 \times p$	Find the discriminant.
$\Rightarrow 9 - 8p > 0 \Rightarrow p < \frac{9}{8}$	Set the discriminant to be greater than 0 and solve the inequality.
b $px^2 + p = 13x \Rightarrow \Delta = (-13)^2 - 4 \times p \times p$ $= 169 - 4p^2$	Find the discriminant.
$\Rightarrow 169 - 4p^2 = 0 \Rightarrow p = \pm\frac{13}{2}$	Set the discriminant to be equal to 0 and solve the equation.
c $(p + 2)x^2 + 2px = 1 - p$	Find the discriminant.
$\Rightarrow \Delta = (2p)^2 - 4(p + 2)(p - 1)$ $= 4p^2 - 4p^2 - 4p + 8$ $= 8 - 4p$	
$\Rightarrow 8 - 4p < 0 \Rightarrow p > 2$	Set the discriminant to be less than 0 and solve the inequality.

Example 8

Find the values of r for which the equation $x^2 + 3rx + 1 = 0$ has

a two distinct real roots **b** one real repeated root **c** no real roots.

$x^2 + 3rx + 1 = 0 \Rightarrow \Delta = (3r)^2 - 4 \times 1 \times 1$ $\Rightarrow \Delta = 9r^2 - 4$	Find the discriminant and simplify it.
a Two distinct real roots:	
$\Delta > 0 \Rightarrow 9r^2 - 4 > 0 \Rightarrow r^2 > \frac{4}{9} \Rightarrow \lvert r\rvert > \frac{2}{3}$	When $\Delta > 0$ there are two distinct real roots. Take the square root of both sides, $\sqrt{r^2} = \lvert r\rvert$. Solve the inequality.
$\Rightarrow r < -\frac{2}{3}$ or $r > \frac{2}{3}$	
b One repeated real root:	When $\Delta = 0$ there is one repeated real root.
$\Delta = 0 \Rightarrow 9r^2 - 4 = 0 \Rightarrow r^2 = \frac{4}{9} \Rightarrow r = \pm\frac{2}{3}$	Solve the equation.
c No real roots:	When $\Delta < 0$ there is no real root.
$\Delta < 0 \Rightarrow 9r^2 - 4 < 0 \Rightarrow r^2 < \frac{4}{9} \Rightarrow \lvert r\rvert < \frac{2}{3}$	Take the square root of both sides, $\sqrt{r^2} = \lvert r\rvert$. Solve the inequality.
$\Rightarrow -\frac{2}{3} < r < \frac{2}{3}$	

All of the inequalities can be solved by graphing on your calculator.



Exercise 3D

1 Without solving these equations, determine the nature of their roots.

a $x^2 + 3x - 7 = 0$ **b** $x^2 + x + 2 = 0$

c $x^2 + 2x + 1 = 0$ **d** $5x^2 + \sqrt{3}x + 2 = 0$

e $2x^2 = \pi x - 1$ **f** $2.25x^2 + 49 = 21x$

2 Find the values of the given parameter for which the equation has

i two distinct real roots

ii one real repeated root

iii no real roots.

a $mx^2 + 2x - 5 = 0$ **b** $4x^2 = 3x + 4 - t$

c $(2s + 1)x^2 = s(3x - 1)$

Solving quadratic inequalities

Apart from using a calculator and identifying the part of the parabola that is above or below the x-axis (shown in Before you start), there are two other methods for solving quadratic inequalities.

Method 1: Algebraic method

In order to solve a quadratic inequality that will factorize, you need to first factorize the quadratic expression into linear factors and then consider the sign of the product. You know that the product of two factors is positive if both factors are of the same sign, and that the product is negative if both factors are of different signs.

$$A \times B > 0 \Rightarrow \begin{cases} A > 0 \text{ and } B > 0 \\ \text{or} \\ A < 0 \text{ and } B < 0 \end{cases} \quad \text{or} \quad A \times B < 0 \Rightarrow \begin{cases} A > 0 \text{ and } B < 0 \\ \text{or} \\ A < 0 \text{ and } B > 0 \end{cases}$$

TOK

We have seen the involvement of several nationalities in the development of quadratics.

To what extent do you believe that mathematics is a product of human social collaboration?

Example 9

Solve the quadratic inequality $2x^2 - 5x - 3 \geq 0$.

$2x^2 - 5x - 3 \geq 0 \Rightarrow 2x^2 - 6x + x - 3 \geq 0$

$\Rightarrow 2x(x - 3) + (x - 3) \geq 0 \Rightarrow (x - 3)(2x + 1) \geq 0$

The quadratic expression will factorize, so split it into two linear factors.

$$\Rightarrow \begin{cases} x - 3 \geq 0 \text{ and } 2x + 1 \geq 0 \\ \text{or} \\ x - 3 \leq 0 \text{ and } 2x + 1 \leq 0 \end{cases}$$

Consider the signs of the product.

$$\Rightarrow \begin{cases} x \geq 3 \text{ and } x \geq -\frac{1}{2} \\ \text{or} \\ x \leq 3 \text{ and } x \leq -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} x \geq 3 \\ \text{or} \\ x \leq -\frac{1}{2} \end{cases}$$

Find the range of values of x which satisfy the inequality.

which can be written as an interval

$x \in \left(-\infty, -\frac{1}{2}\right] \cup [3, \infty)$

Method 2: Using a "sign table"

If a quadratic expression will not factorize, then you cannot split it into two linear terms to compare signs of factors.

If this is the case, it is best to find any zeros of the quadratic by solving the corresponding quadratic equation, and then consider the signs of the quadratic either side of the zeros.

This method will work whether or not the quadratic will factorize.

Example 10

Solve the quadratic inequality $2x^2 - 5x - 3 \geq 0$.

$2x^2 - 5x - 3 = 0 \Rightarrow x = \dfrac{5 \pm \sqrt{25 + 24}}{4} = \dfrac{5 \pm 7}{4}$

$x = 3$ or $x = -\dfrac{1}{2}$

Solve the corresponding quadratic equation, eg by using the formula.

	$x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$-\frac{1}{2} < x < 3$	$x = 3$	$x > 3$
$2x^2 - 5x - 3$	+	0	−	0	+

Order the zeros in the table and investigate the sign of the quadratic between the zeros.

$x \in \left(-\infty, -\dfrac{1}{2}\right] \cup [3, \infty)$

The coefficient of x^2 is positive so the parabola is concave up, so between the zeros the expression has a negative value.

Since the inequality is not strict, you should include the zeros in your solution.

Exercise 3E

Solve the following quadratic inequalities by the method of your choice and verify your solutions by a graphical method on a calculator.

1 $x^2 + 6x + 8 < 0$

2 $x^2 - 8x + 16 > 0$

3 $x^2 - 13x - 30 \geq 0$

4 $4x^2 - 11x + 6 \leq 0$

5 $8 < 5x^2 - 6x$

6 $12x - 4 \geq 9x^2$

Developing inquiry skills

Return to the opening problem.

Can you use a quadratic equation to model the roller coaster?



3.2 Complex numbers

Algebraic vs. geometric approach towards complex numbers

Investigation 4

Consider the quadratic equation $x^2 + 4 = 0$.

1 Can you find real values that are solutions of the equation? Explain why, or why not.

Mathematicians define the imaginary number i such that $i^2 = -1$, therefore you can write $i = \sqrt{-1}$.

2 Can you use this fact to solve the quadratic equation, expressing your answers in terms of i?

3 Use the quadratic formula to solve the equation $x^2 - 2x + 5 = 0$. Give the roots in terms of i.

4 **Conceptual** What does the imaginary number i represent and how is it used in solving quadratics?

TOK

Descartes showed that geometric problems could be solved algebraically and vice versa.

What does this tell us about mathematical representation and mathematical knowledge?

HINT

Recall that

$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

$a, b \geq 0$

The letter i is used to represent the imaginary number $\sqrt{-1}$.

International-mindedness

Greek mathematician Heron discussed the root of a negative in the first century BC.

Solutions of any equation $x^2 + c = 0$, $c > 0$, can be written as $x = \pm\sqrt{-c} = \pm i d$, $d \in \mathbb{R}^+$. We call these **imaginary numbers**.

By defining $i^2 = -1$, you have developed a whole new avenue of algebra that you are going to use in this chapter.

Complex numbers are numbers of the form $z = a + ib$, where $a, b \in \mathbb{R}$.

a is called the **real part** of z, and we write $\mathrm{Re}(z) = a$.

b is called the **imaginary part** of z, and we write $\mathrm{Im}(z) = b$.

Complex numbers can be written in different forms, as you will see later in the book. The form $z = a + ib$ is called the **Cartesian form** of a complex number.

Complex numbers were first introduced when mathematicians were trying to solve certain cubic equations. However, as you have seen in the above investigation, complex numbers also allow you to find solutions to any quadratic equation, regardless of the value of its discriminant.

Did you know?

The first person to propose working with the square root of a negative number was Heron of Alexandria (c.10–c.60) when discussing the volume of an impossible frustum of a pyramid.

Later on, in Italy, mathematical tournaments became very popular and solving difficult cubic equations became a way in which they found the winner. Those mathematicians discovered the formula for solutions of cubic equations and introduced complex numbers. Niccolo Fontana Tartaglia (1499–1557) discovered the formula first and shared his knowledge with Gerolamo Cardano (1501–1576).

Cardano introduced complex numbers of the form $a+\sqrt{-b}$, $a\in\mathbb{R}$, $b\in\mathbb{R}^+$. They realized that those two parts cannot be combined and the second part was called the imaginary or even impossible part.

René Descartes (1596–1650) was the first mathematician to establish the term "imaginary part", and John Wallis (1616–1703) made huge progress in giving a geometrical interpretation to $\sqrt{-1}$.
Leonhard Euler (1707–1783) was the mathematician who first introduced the symbol $\mathrm{i}=\sqrt{-1}$ and called it an "imaginary unit".

By looking at the complex number $z=a+\mathrm{i}b$, you might notice that every real number a can be seen as a complex number where the imaginary part is 0, that is, $b=0$. In a similar way, a complex number that has real part equal to 0, ie $a=0$, only has an imaginary part and is called a purely imaginary number. The number 0 can be seen as both, $0=0+\mathrm{i}\times 0$.

In this way, you can view the real numbers as a subset of the complex numbers, and hence you can expand the subset definition from Chapter 1: $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$.

Example 11

State the real part and the imaginary part of the following complex numbers.

a $2+3\mathrm{i}$ **b** $2\mathrm{i}-5$ **c** $\frac{2}{3}-\sqrt{7}\mathrm{i}$ **d** $\frac{\sqrt{11}+\pi\mathrm{i}}{4}$

a $z=2+3\mathrm{i}\Rightarrow \mathrm{Re}(z)=2$ and $\mathrm{Im}(z)=3$ **b** $z=2\mathrm{i}-5\Rightarrow \mathrm{Re}(z)=-5$ and $\mathrm{Im}(z)=2$ **c** $z=\frac{2}{3}-\sqrt{7}\mathrm{i}\Rightarrow \mathrm{Re}(z)=\frac{2}{3}$ and $\mathrm{Im}(z)=-\sqrt{7}$ **d** $z=\frac{\sqrt{11}+\pi\mathrm{i}}{4}\Rightarrow \mathrm{Re}(z)=\frac{\sqrt{11}}{4}$ and $\mathrm{Im}(z)=\frac{\pi}{4}$	The real part has no imaginary unit and the imaginary part has an imaginary unit.

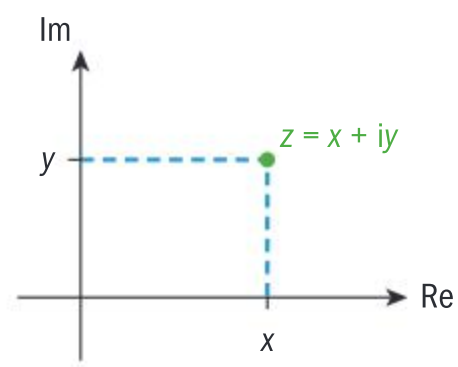


Geometric approach

Real numbers can be visualized on a number line. Each point on the line represents one real number.

In a similar way, complex numbers can be represented in a two-dimensional coordinate plane, where the horizontal axis represents the real part of the number, and the vertical axis represents the imaginary part of the number.

Each complex number $z = x + iy$, $x, y \in \mathbb{R}$, is represented by a point $P(x, y)$ in the plane and the coordinates are the real and imaginary parts of the complex number itself.



Purely real numbers lie on the x-axis, and purely imaginary numbers lie on the y-axis.

Modulus or absolute value of a complex number

You already know that the modulus or absolute value of a real number is algebraically defined as $|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$. Geometrically, the modulus represents the distance along the number line between x and the origin. You can extend this idea into two dimensions in the complex plane where the modulus of a complex number represents the distance between the complex number and the origin.

Investigation 5

1 Plot the points that represent the following complex numbers on a complex plane. $3+4i, 1-2i, -5+4i, -2, 3i, -2-3i$

2 What is the distance between each point you plotted in question **1** and the origin?

3 **Factual** Can you write down a formula for finding the modulus, $|z|$, of a complex number $z = x + iy$?

4 **Factual** What do you notice about the modulus of a complex number?

5 **Conceptual** What does the modulus of a complex number tell you about that number?

Given the complex number $z = x + iy$, $x, y \in \mathbb{R}$, the **modulus** of z is given by

$$|z| = |x + iy| = \sqrt{x^2 + y^2} = \sqrt{\left(\mathrm{Re}(z)\right)^2 + \left(\mathrm{Im}(z)\right)^2}.$$

Notice that the modulus of a complex number is always a positive real number, $|z| \in \mathbb{R}^+$.

Did you know?

Jean-Robert Argand (1768–1822) first proposed the idea of representing complex numbers on a two-dimensional coordinate plane. Carl Friedrich Gauss (1777–1855) independently developed and refined the plane model and therefore the plane of complex numbers is known as the **Argand diagram** or **Gaussian plane**. Caspar Wessel (1745–1818) also developed a similar approach using vectors.

Example 12

Find the modulus of the following complex numbers.

a $5-12\mathrm{i}$ **b** $\frac{1}{2}-\frac{\mathrm{i}}{4}$ **c** $\frac{20\mathrm{i}-21}{29}$

Check your answers with a GDC.

a $|5-12\mathrm{i}| = \sqrt{5^2+(-12)^2} = \sqrt{169} = 13$

Apply the formula for the modulus.

b $\left|\frac{1}{2}-\frac{\mathrm{i}}{4}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2} = \sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$

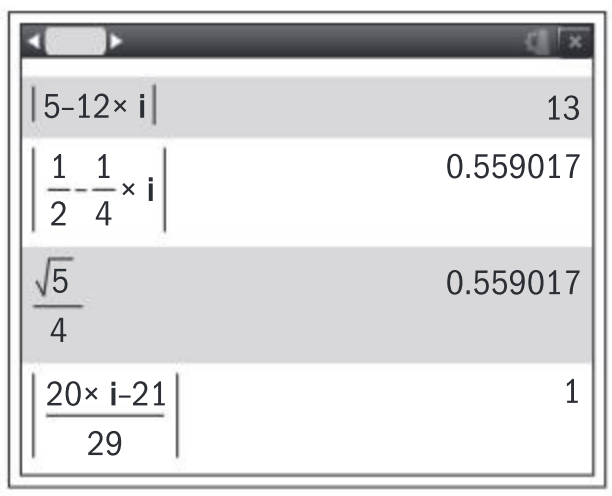
Apply the formula for the modulus.

c $\left|\frac{20\mathrm{i}-21}{29}\right| = \sqrt{\left(-\frac{21}{29}\right)^2+\left(\frac{20}{29}\right)^2}$

$= \sqrt{\frac{841}{841}} = 1$

Apply the formula for the modulus.

Checking:



Make sure you know how to use a calculator to find the modulus of a complex number. You can use this to check the results.

Operations with complex numbers

First you need to understand when two complex numbers are equal.

Two complex numbers $z_1 = a+b\mathrm{i}$ and $z_2 = c+d\mathrm{i}$, $a, b, c, d \in \mathbb{R}$, are equal if and only if their real parts are equal and their imaginary parts are equal, $a = c$ and $b = d$.

Investigation 6

Use your calculator to complete the first row of this table. z_1 and z_2 are complex numbers, and m and n are real constants.

Then add your own examples: for each blank row, define any two complex numbers z_1 and z_2, and any two real constants m and n, and complete the table.



z_1	z_2	$z_1 + z_2$	m	$m \times z_1$	n	$n \times z_2$	$m \times z_1 + n \times z_2$
$z_1 = 2 + \mathrm{i}$	$z_2 = 3 - 2\mathrm{i}$		2		-3		

Use the table you just completed to answer the following questions.

1 **Factual** How do you add two complex numbers?

2 **Factual** How do you subtract two complex numbers?

3 **Conceptual** Do we need two separate operations for adding and subtracting complex numbers?

4 **Factual** How do you multiply (or divide) a complex number by a real number?

5 **Conceptual** How are algebraic operations with complex numbers similar to algebraic operations with polynomials?

When adding complex numbers, or multiplying them by a constant, use the following rules.

$$z_1 + z_2 = (a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}, a, b, c, d \in \mathbb{R}$$
$$\lambda z = \lambda(a + b\mathrm{i}) = (\lambda a) + (\lambda b)\mathrm{i}, \lambda, a, b \in \mathbb{R}$$

International-mindedness

How do you use the Babylonian method of multiplication?

Try 36×14.

Example 13

If $z_1 = 4 - 3\mathrm{i}$ and $z_2 = -2 + 5\mathrm{i}$, calculate the following and check your answers with a calculator.

a $z_1 - z_2$ **b** $\frac{1}{3}z_1 + \frac{2}{5}z_2$

a $z_1 - z_2 = (4 - 3\mathrm{i}) - (-2 + 5\mathrm{i})$
$= (4 + 2) + (-3 - 5)\mathrm{i} = 6 - 8\mathrm{i}$

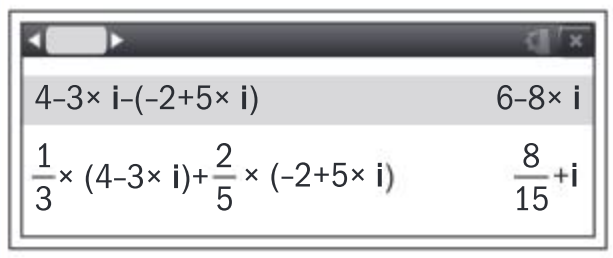
Use the properties of complex numbers.

b $\frac{1}{3}z_1 + \frac{2}{5}z_2 = \frac{1}{3}(4 - 3\mathrm{i}) + \frac{2}{5}(-2 + 5\mathrm{i})$

$= \frac{4}{3} - \mathrm{i} - \frac{4}{5} + 2\mathrm{i} = \frac{8}{15} + \mathrm{i}$

Multiply both z_1 and z_2 by the given constant, and then add.

Checking:



Make sure you know how to use a calculator to add and subtract complex numbers and to multiply complex numbers by a constant. You can use this to check the results.

Exercise 3F

1 Find the real and imaginary parts of the following complex numbers.

a $z = -4\text{i}$ **b** $z = 5$

c $z = -24 + 7\text{i}$ **d** $z = \dfrac{5 - 12\text{i}}{13}$

e $z = \dfrac{2\text{i} - 1}{\sqrt{5}}$

2 Find the modulus of each of the complex numbers in question **1**.

3 Given the complex numbers $z_1 = 3 - 4\text{i}$, $z_2 = 5 + \text{i}$ and $z_3 = -1 + 2\text{i}$, calculate the following and check your answers with a calculator.

a $z_1 - z_2 + z_3$ **b** $2z_1 + 3z_2 - 4z_3$

c $-\dfrac{1}{2}z_1 + \dfrac{2}{3}z_2 - \dfrac{1}{4}z_3$ **d** $\dfrac{4z_3 - 5z_1 + 2z_2}{3}$

Investigation 7

Use your calculator and your algebra skills to fill in the following table. The first line has been done for you.

z_1	z_2	$z_1 \times z_2$
$z_1 = 2 + \text{i}$	$z_2 = 1 + 3\text{i}$	$(2 + \text{i}) \times (1 + 3\text{i}) = 2 + \text{i} + 6\text{i} + \underbrace{3\text{i}^2}_{-1} = -1 + 7\text{i}$
$z_1 = 1 - 2\text{i}$	$z_2 = 3 + \text{i}$	
$z_1 = 3 + 2\text{i}$	$z_2 = 4 - 5\text{i}$	
$z_1 = a + b\text{i}$	$z_2 = 3 + \text{i}$	
$z_1 = 3 + 2\text{i}$	$z_2 = c + d\text{i}$	
$z_1 = a + b\text{i}$	$z_2 = c + d\text{i}$	

1 **Factual** Using words, express how the real part of $z_1 \times z_2$ is related to the real and imaginary parts of both z_1 and z_2.

2 **Factual** In a similar way, express how the imaginary part of $z_1 \times z_2$ is related to the real and imaginary parts of both z_1 and z_2.

3 **Conceptual** How would you compare multiplication of two complex numbers to multiplication of two algebraic linear factors?

To summarize the findings from the investigation, we write the following.

$$z_1 \times z_2 = (a + b\text{i}) \times (c + d\text{i}) = (ac - bd) + (ad + bc)\text{i},\ a, b, c, d \in \mathbb{R}$$

Although the formula in the key point above gives you a general rule for multiplying two complex numbers, in reality this rule does not simplify the process of multiplication much. Therefore, when asked to multiply two complex numbers, it is often simpler to use algebra to find their product.



Example 14



Given the complex numbers $z_1 = 1 - 3\mathrm{i}$, $z_2 = 4 + \mathrm{i}$ and $z_3 = -2 + 3\mathrm{i}$, find the following.

a $z_1 \times z_2 - z_3$ **b** $z_1 \times z_2 \times z_3$ **c** $z^2_1 + 2z_2 \times z_3$

Use technology to check your answer.

a $z_1 \times z_2 - z_3 = (1-3\mathrm{i})(4+\mathrm{i}) - (-2+3\mathrm{i})$
$= 4 + \mathrm{i} - 12\mathrm{i} + 3 + 2 - 3\mathrm{i}$
$= 9 - 14\mathrm{i}$

Expand the brackets and simplify the expression.

b $z_1 \times z_2 \times z_3 = (1-3\mathrm{i})(4+\mathrm{i})(-2+3\mathrm{i})$
$= (7 - 11\mathrm{i})(-2+3\mathrm{i})$
$= -14 + 21\mathrm{i} + 22\mathrm{i} + 33$
$= 19 + 43\mathrm{i}$

First multiply the first two factors, then multiply the product with the third factor.

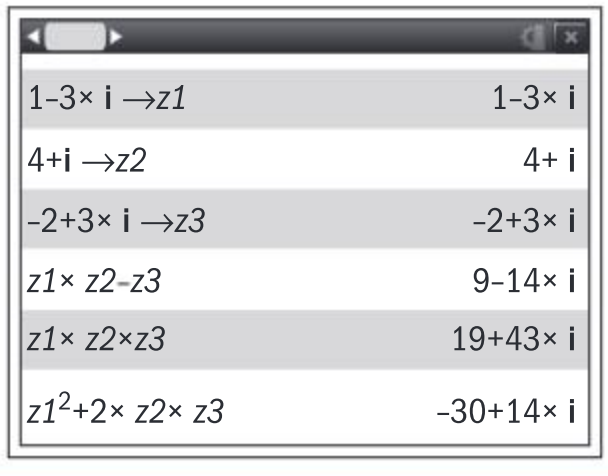
Simplify the expression.

c $z^2_1 + 2z_2 \times z_3 = (1-3\mathrm{i})^2 + 2(4+\mathrm{i})(-2+3\mathrm{i})$
$= 1 - 6\mathrm{i} - 9 + 2(-8 + 12\mathrm{i} - 2\mathrm{i} - 3)$
$= -8 - 6\mathrm{i} - 22 + 20\mathrm{i}$
$= -30 + 14\mathrm{i}$

Square z_1 and multiply z_2 by z_3.

Expand the expressions and simplify.

Checking:



1-3× i →z1	1-3× i
4+i →z2	4+ i
-2+3× i →z3	-2+3× i
z1× z2-z3	9-14× i
z1× z2×z3	19+43× i
$z1^2$+2× z2× z3	-30+14× i

Make sure you know how to use a calculator to multiply two complex numbers. You can use this to check the results. For ease, store the values of z_1, z_2 and z_3 in the GDC memory and then work with the variables.

Example 15

Find $z \in \mathbb{C}$ that satisfies the equation $z \times (2 - \mathrm{i}) = z + 3 + 2\mathrm{i}$

Use technology to check your answer.

Let $z = a + b\mathrm{i}$

$\Rightarrow (a + b\mathrm{i}) \times (2 - \mathrm{i}) = (a + b\mathrm{i}) + 3 + 2\mathrm{i}$

$\Rightarrow (2a + b) + (-a + 2b)\mathrm{i} = (a + 3) + (b + 2)\mathrm{i}$

$\Rightarrow \begin{cases} 2a + b = a + 3 \\ -a + 2b = b + 2 \end{cases} \Rightarrow \begin{cases} a + b = 3 \\ -a + b = 2 \end{cases}$

Rewrite z in Cartesian form and substitute it into the equation.

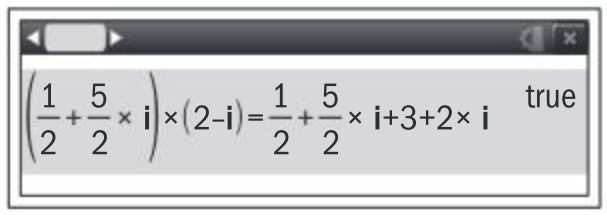
Simplify both sides of the equation, and collect real and imaginary parts.

If two complex numbers are equal then both the real parts are equal and the imaginary parts are equal.

Continued on next page

$$\Rightarrow (a, b) = \left(\frac{1}{2}, \frac{5}{2}\right) \Rightarrow z = \frac{1}{2} + \frac{5}{2}\text{i}$$

Checking:



Simplify and solve the simultaneous equations.

Use your calculator to check the solution.

Conjugate complex numbers

For any complex number $z = a + b\text{i}$, $a, b \in \mathbb{R}$, there is a **conjugate** complex number of the form $z^* = a - b\text{i}$. Their real parts are equal, $\text{Re}(z) = \text{Re}(z^*)$, and their imaginary parts are opposite, $\text{Im}(z) = -\text{Im}(z^*)$.

Notice that when adding two conjugate complex numbers you get **twice the real part**, that is $z + z^* = a + b\text{i} + a - b\text{i} = 2a$.

Also, when subtracting two conjugate complex numbers you get **twice the imaginary part**, multiplied by i, that is $z - z^* = a + b\text{i} - (a - b\text{i}) = 2b\text{i}$.

Investigation 8

Copy and complete the following table. The first row has been done for you. Use a GDC to check your answers.

z	$\lvert z\rvert$	z^*	$z \times z^*$
$z = -1 + \text{i}$	$\lvert z\rvert = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$	$z^* = -1 - \text{i}$	$z \times z^* = 2$
$z = 3 - 4\text{i}$			
$z = -\sqrt{5} + 2\text{i}$			
$z = a + b\text{i}$			

1 **Factual** What is the relationship between the product and the modulus?

2 **Conceptual** Why do a complex number and its conjugate have equal moduli?

3 **Conceptual** Verify that z and z^* are mutually conjugate.



Division of complex numbers

In Investigation 6 and Example 13, you saw how to divide a complex number by a real number. The following example demonstrates how the complex conjugate helps us to divide two complex numbers.

Example 16

Given two complex numbers $z_1 = a + b\mathrm{i}$ and $z_2 = c + d\mathrm{i}$, $a, b, c, d \in \mathbb{R}, z_2 \neq 0$, find $\frac{z_1}{z_2}$.

$\frac{z_1}{z_2} = \frac{a+b\mathrm{i}}{c+d\mathrm{i}} = \frac{a+b\mathrm{i}}{c+d\mathrm{i}} \times \frac{c-d\mathrm{i}}{c-d\mathrm{i}}$	Multiply numerator and denominator by the complex conjugate of the denominator.
$= \frac{ac - ad\mathrm{i} + bc\mathrm{i} - bd\overbrace{\mathrm{i}^2}^{-1}}{c^2 + d^2}$	The denominator is then a real number, and you know how to divide a complex number by a real number.
$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\mathrm{i}$	Simplify the numerator, and split into real and imaginary parts.

Notice that, since $z \times z^* = |z|^2$, we can write $\frac{z_1}{z_2} = \frac{z_1 \times z_2^*}{|z_2|^2}$.

Example 17



If $z_1 = 2 + \mathrm{i}$, $z_2 = 2 - 5\mathrm{i}$ and $z_3 = -1 + 2\mathrm{i}$, find the following:

a $\frac{z_1}{z_2} - 3z_3$ **b** $\frac{z_1^{\,2}}{z_2 \times z_3}$ **c** $\frac{2z_1 - 3z_2}{z_1 \times z_3}$

Check your answers with a calculator.

a

$\frac{z_1}{z_2} - 3z_3 = \frac{2+\mathrm{i}}{2-5\mathrm{i}} \times \frac{2+5\mathrm{i}}{2+5\mathrm{i}} - 3(-1+2\mathrm{i})$	Multiply both numerator and denominator by the conjugate of the denominator.
$= \frac{4+10\mathrm{i}+2\mathrm{i}-5}{4+25} + 3 - 6\mathrm{i}$	Expand and simplify.
$= \frac{-1+12\mathrm{i}+87-174\mathrm{i}}{29}$	Write the expression over a common denominator, and simplify.
$= \frac{86-162\mathrm{i}}{29}$	

Continued on next page

b $\dfrac{z_1^{\,2}}{z_2 \times z_3} = \dfrac{(2+\mathrm{i})^2}{(2-5\mathrm{i})(-1+2\mathrm{i})}$

$= \dfrac{4+4\mathrm{i}-1}{-2+4\mathrm{i}+5\mathrm{i}+10}$

$= \dfrac{3+4\mathrm{i}}{8+9\mathrm{i}}$

$= \dfrac{(3+4\mathrm{i})(8-9\mathrm{i})}{|8+9\mathrm{i}|^2}$

Expand the numerator and multiply the factors in the denominator.

You could multiply top and bottom by the conjugate of the bottom. Here, we have used $\dfrac{z_1}{z_2} = \dfrac{z_1 \times z_2^*}{|z_2|^2}$.

$= \dfrac{24-27\mathrm{i}+32\mathrm{i}+36}{64+81}$

$= \dfrac{60+5\mathrm{i}}{145} = \dfrac{12+\mathrm{i}}{29}$

Multiply out both numerator and denominator.

Simplify the answer.

c $\dfrac{2z_1 - 3z_2}{z_1 \times z_3} = \dfrac{2(2+\mathrm{i})-3(2-5\mathrm{i})}{(2+\mathrm{i})(-1+2\mathrm{i})}$

$= \dfrac{4+2\mathrm{i}-6+15\mathrm{i}}{-2+4\mathrm{i}-\mathrm{i}-2}$

$= \dfrac{-2+17\mathrm{i}}{-4+3\mathrm{i}}$

$= \dfrac{(-2+17\mathrm{i})(-4-3\mathrm{i})}{|-4+3\mathrm{i}|^2}$

$= \dfrac{8+6\mathrm{i}-68\mathrm{i}+51}{16+9} = \dfrac{59-62\mathrm{i}}{25}$

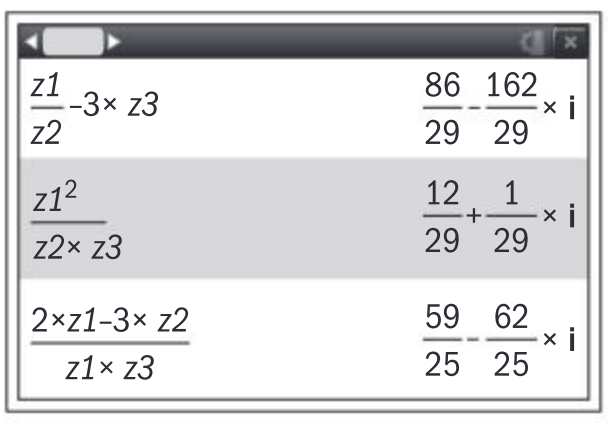
Expand the numerator and multiply the factors in the denominator.

Simplify.

Using $\dfrac{z_1}{z_2} = \dfrac{z_1 \times z_2^*}{|z_2|^2}$

Multiply the factors in the numerator and simplify the expression.

Checking:



Use a calculator to check the results. For ease, store all three values in the memory and then work with the variables.

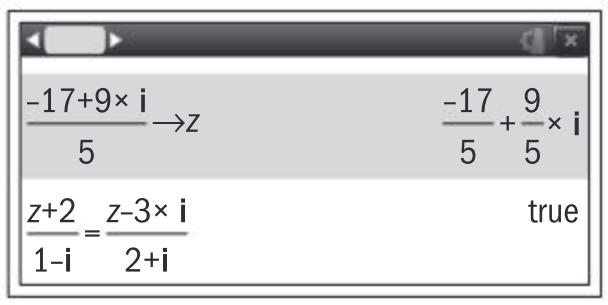
Example 18

Find $z \in \mathbb{C}$ that satisfies the equation $\dfrac{z+2}{1-\mathrm{i}} = \dfrac{z-3\mathrm{i}}{2+\mathrm{i}}$

Use your calculator to check the solution.



$\frac{z+2}{1-\mathrm{i}} = \frac{z-3\mathrm{i}}{2+\mathrm{i}}$	Multiply both sides of the equation by $(1-\mathrm{i})(2+\mathrm{i})$.
$\Rightarrow (z+2)\times(2+\mathrm{i}) = (z-3\mathrm{i})\times(1-\mathrm{i})$	
$\Rightarrow 2z + z\mathrm{i} + 4 + 2\mathrm{i} = z - z\mathrm{i} - 3\mathrm{i} - 3$	Multiply out.
$\Rightarrow 2z - z + z\mathrm{i} + z\mathrm{i} = -3\mathrm{i} - 3 - 4 - 2\mathrm{i}$	Isolate the terms in z on one side of the equation.
$\Rightarrow z + 2z\mathrm{i} = -7 - 5\mathrm{i}$	Simplify.
$\Rightarrow z(1+2\mathrm{i}) = -7 - 5\mathrm{i}$	Factorize.
$z = \frac{-7-5\mathrm{i}}{1+2\mathrm{i}}$	Make z the subject.
$= \frac{(-7-5\mathrm{i})(1-2\mathrm{i})}{\lvert 1+2\mathrm{i}\rvert^2}$	Using $\frac{z_1}{z_2} = \frac{z_1 \times z_2^*}{\lvert z_2\rvert^2}$
$= \frac{-7-5\mathrm{i}+14\mathrm{i}-10}{1+4}$	
$= \frac{-17+9\mathrm{i}}{5}$	
Checking:	Use your calculator to check the solution.



Exercise 3G

1 Given the complex numbers $z_1 = 1 + \mathrm{i}$, $z_2 = 3 - 2\mathrm{i}$ and $z_3 = -2 + 3\mathrm{i}$, calculate the following and check your answers with a calculator.

a $z_1 \times z_2 + 2z_3^*$ **b** $\frac{z_1}{z_3} - \frac{z_2}{5}$

c $z_1^2 - 3z_2 z_3$ **d** $\frac{z_1 z_3}{z_2^*}$

e $\frac{2z_1 - 4z_2^*}{z_3\ z_2^*}$

2 Find the real and imaginary parts of the following complex numbers.

a $\frac{1+2\mathrm{i}}{\mathrm{i}}$ **b** $\frac{1}{\mathrm{i}} + \frac{2\mathrm{i}}{1-\mathrm{i}}$

c $\frac{1+2\mathrm{i}}{1-2\mathrm{i}} - \frac{1-2\mathrm{i}}{1+2\mathrm{i}}$

3 Find the real numbers a and b that satisfy the following:

a $(1+3\mathrm{i})(a+b\mathrm{i}) = 5 + 5\mathrm{i}$

b $\frac{a+b\mathrm{i}}{1+2\mathrm{i}} = -3 + \mathrm{i}$

4 Find the complex number z that satisfies the following equations.

a $2(z+\mathrm{i}) = 3\mathrm{i}(z-1)$

b $\frac{z-2}{1-2\mathrm{i}} = \frac{z-\mathrm{i}}{2+\mathrm{i}}$

c $(z+2\mathrm{i})(2+\mathrm{i}) = (z-1)(1-\mathrm{i})$

d $\frac{z+1+\mathrm{i}}{1-4\mathrm{i}} = \frac{z-3\mathrm{i}+2}{2\mathrm{i}+5}$

5 Given the complex number $z = a + b\mathrm{i}$, a, $b \in \mathbb{R}$, find the relationship between a and b such that

a $\dfrac{z}{2+\mathrm{i}} \in \mathbb{R}$ **b** $\dfrac{1-\mathrm{i}}{z^*}$ is purely imaginary.

6 Solve for $z \in \mathbb{C}$.

a $|z| + z = 1$

b $|z| - z^* = \mathrm{i}$

c $z^2 + z^* = 2$

7 Prove the following properties of the modulus of complex numbers.

a $|z_1 \times z_2| = |z_1| \times |z_2|$ **b** $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

c $|z_1 + z_2| \leq |z_1| + |z_2|$

8 Prove the following properties of conjugate complex numbers.

a $\left(z^*\right)^* = z$

b $(z_1 + z_2)^* = {z_1}^* + {z_2}^*$

c $(z_1 \times z_2)^* = {z_1}^* \times {z_2}^*$

d $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$

e $|z| = |z^*|$

Powers and roots of complex numbers

In order to raise a complex number to a power, you first need to investigate what happens when you raise the imaginary unit, i, to a power.

TOK

Could we ever reach a point where everything important in a mathematical sense is known?

Investigation 9

Fill in the following table by calculating the powers of i.

n	0	1	2	3	4	5	6	7	8	9
i^n	1	i								

1 Describe a pattern that your table shows.

2 Find a general rule for i^n, $n \in \mathbb{N}$.

3 Use your general rule to find i^{2019}.

4 Does your rule also apply for negative powers of i, that is, when $n \in \mathbb{Z}^-$?

5 **Conceptual** Can you generalize the powers of imaginary numbers?

Now you will use your general rule to raise any complex number to an integer power.

6 Let $z = a + b\mathrm{i}$. In Chapter 1 you learned the binomial expansion of $(a+b)^n$, $n \in \mathbb{N}$. Use the binomial expansion, and your results about powers of i from question **2**, to find expressions for z^2 and z^3.

7 Use the same method to find $(z^4)^*$ and $(z^*)^4$. What do you notice?



The following example uses some higher powers of a complex number.

Example 19



Given the complex number $z = 2 + \mathrm{i}$, find each of these expressions. Check your answer with a calculator.

a z^3 **b** $\left(z^*\right)^5$ **c** $\left(z^5\right)^*$

a $z^3 = (2 + \mathrm{i})^3$

$= 2^3 + 3 \times 2^2 \times \mathrm{i} + 3 \times 2 \times \mathrm{i}^2 + \mathrm{i}^3$

$= 8 + 12\mathrm{i} - 6 - \mathrm{i} = 2 + 11\mathrm{i}$

Use the binomial theorem for $n = 3$.

Use the powers of i. Simplify the expression.

b $\left(z^*\right)^5 = (2 - \mathrm{i})^5$

$= 2^5 + 5 \times 2^4(-\mathrm{i}) + 10 \times 2^3(-\mathrm{i})^2$

$+ 10 \times 2^2(-\mathrm{i})^3 + 5 \times 2(-\mathrm{i})^4 + (-\mathrm{i})^5$

$= 32 - 80\mathrm{i} - 80 + 40\mathrm{i} + 10 - \mathrm{i} = -38 - 41\mathrm{i}$

Use the binomial theorem for $n = 5$.

Use the powers of i. Simplify the expression.

c First calculate z^5 and then $\left(z^5\right)^*$.

$z^5 = (2 + \mathrm{i})^5$

$= 2^5 + 5 \times 2^4\mathrm{i} + 10 \times 2^3\mathrm{i}^2 + 10 \times 2^2\mathrm{i}^3 + 5 \times 2\mathrm{i}^4 + \mathrm{i}^5$

$= 32 + 80\mathrm{i} - 80 - 40\mathrm{i} + 10 + \mathrm{i} = -38 + 41\mathrm{i}$

$\Rightarrow \left(z^5\right)^* = (-38 + 41\mathrm{i})^* = -38 - 41\mathrm{i}$

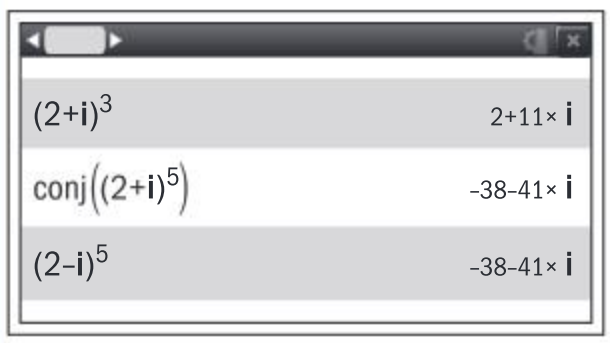
Use the binomial theorem for $n = 5$.

Use the powers of i. Simplify the expression.

Find the conjugate complex number.

Use a calculator to check the solutions.

Checking:



You can find square roots of complex numbers algebraically by letting the root be equal to a complex number $a + \mathrm{i}b$. You then square this, and equate real and imaginary parts.

You can use a similar technique for finding higher roots of complex numbers, such as cube roots or fourth roots.

Example 20 will show you how to do this.

Example 20

Evaluate $\sqrt{7-24\mathrm{i}}$.

Let $z = a + b\mathrm{i}, a, b \in \mathbb{R}$	
$z = \sqrt{7-24\mathrm{i}} \Rightarrow (a+b\mathrm{i})^2 = 7-24\mathrm{i}$	Write z in Cartesian form.
$\Rightarrow a^2 + 2ab\mathrm{i} - b^2 = 7 - 24\mathrm{i}$	Square both sides and expand the product.
$\Rightarrow \begin{cases} a^2 - b^2 = 7 \\ 2ab = -24 \end{cases} \Rightarrow \begin{cases} a^2 - \dfrac{144}{a^2} = 7 \\ b = -\dfrac{12}{a} \end{cases}$	Equate real parts and imaginary parts.
$\Rightarrow \begin{cases} a^4 - 7a^2 - 144 = 0 \\ b = -\dfrac{12}{a} \end{cases} \Rightarrow \begin{cases} (a^2+9)(a^2-16) = 0 \\ b = -\dfrac{12}{a} \end{cases}$	Solve simultaneous equations by using the method of substitution. Factorize the expression. Since a is real, we discard $(a^2 + 9) = 0$.
$\Rightarrow \begin{cases} a^2 = 16 \\ b = -\dfrac{12}{a} \end{cases} \Rightarrow \begin{cases} a = \pm 4 \\ b = -\dfrac{12}{\pm 4} = \mp 3 \end{cases}$	For two values of a find two values of b.
$z = 4 - 3\mathrm{i}$ or $z = -4 + 3\mathrm{i}$	Find both solutions.

A calculator can also be used to evaluate this square root, but the calculator will only give you one solution.



You have to know that the second solution is the opposite number, that is $-4 + 3\mathrm{i}$.

As you saw in Example 20, finding square roots of complex numbers requires algebraic skill in solving simultaneous equations of degree two or higher.

Exercise 3H

1 Calculate

a $\mathrm{i}^7 + \mathrm{i}^{17} + \mathrm{i}^{27} + \mathrm{i}^{37}$

b $\mathrm{i}^{173} - \mathrm{i}^{272} + \mathrm{i}^{351} - \mathrm{i}^{766}$

c $(3 + \mathrm{i}^{77}) \times (1 - 2\mathrm{i}^{93})$

d $\dfrac{3\mathrm{i}^{2018} + 2\mathrm{i}^{2019}}{4\mathrm{i}^{2020} - 3\mathrm{i}^{2021}}$

e $\dfrac{\sum_{k=1}^{2019} \mathrm{i}^k}{\prod_{k=1}^{2019} \mathrm{i}^k}$

f $\dfrac{\sum_{k=1}^{1010} \mathrm{i}^{2k-1}}{\prod_{k=1}^{1010} \mathrm{i}^{2k}}$

HINT

Similar to Σ-notation for a sum there is a product notation

$$\prod_{k=1}^{n} \mathrm{i}^k = \mathrm{i}^1 \times \mathrm{i}^2 \times \ldots \times \mathrm{i}^n$$



2 Calculate the following and check your answers with a calculator.

a $(3 + 2\mathrm{i})^3$

b $(1 - 3\mathrm{i})^4$

c $(1 - 2\mathrm{i})^4 + (1 + 2\mathrm{i})^4$

d $(1 + \mathrm{i})^5 + (1 - \mathrm{i})^5$

3 Evaluate the following and check your answers with a calculator.

a $\sqrt{\mathrm{i}}$ **b** $\sqrt{-\mathrm{i}}$

c $\sqrt{-21+20\mathrm{i}}$ **d** $\sqrt{\frac{5}{36}-\frac{\mathrm{i}}{3}}$

4 Show that

a $|z^n| = |z|^n$, $n \in \mathbb{Z}^+$

b $(z*)^n = (z^n)^*$, $n \in \mathbb{Z}^+$

5 Given that $z = 1 + \mathrm{i}$, find the values of $n \in \mathbb{Z}^+$ such that

a z^n is real

b z^n is purely imaginary.

6 Show that

a $(1 - \mathrm{i})^{2n} = (-2\mathrm{i})^n$, $n \in \mathbb{N}$

b $(1 - \mathrm{i})^{2n+1} = (1 - \mathrm{i})(-2\mathrm{i})^n$, $n \in \mathbb{N}$

3.3 Polynomial equations and inequalities

Investigation 10

1 Draw the graphs of $f_n(x) = x^n$, $n \in \mathbb{N}$, up to at least $n = 7$.

2 **Conceptual** How could you classify these graphs by their shapes?

3 **Conceptual** How could you classify these graphs by their symmetrical properties?

Polynomials are functions which map a real variable, often called x, to another real number. We write $f: \mathbb{R} \to \mathbb{R}$.

Polynomials are functions of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_i \in \mathbb{R}$, $i = 0, \ldots, n$ are called the **coefficients.** The highest power (n) of the variable x is called the **degree** of the polynomial, and we write $\deg(f) = n$.

A **linear combination** of two functions f and g is an expression of the form $a \times f(x) + b \times g(x)$, where a and b are some real numbers.

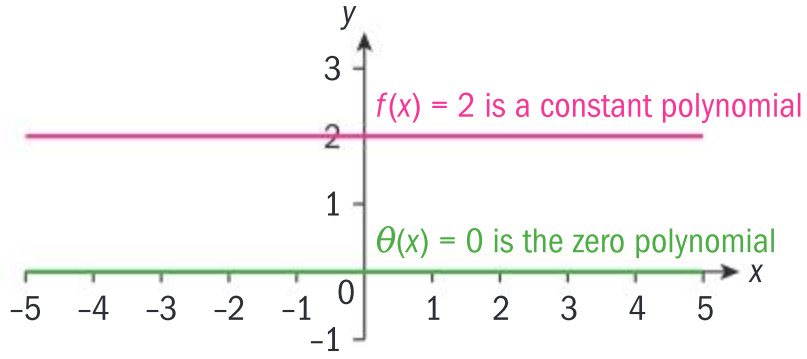
A linear combination of n functions is an expression of the form $\sum_{i=1}^{n} a_i \times f_i(x)$, where f_i are functions and $a_i \in \mathbb{R}$.

In general, you can say that polynomials are linear combinations of the power functions $\{1, x, x^2, x^3, x^4, x^5, \ldots\}$.

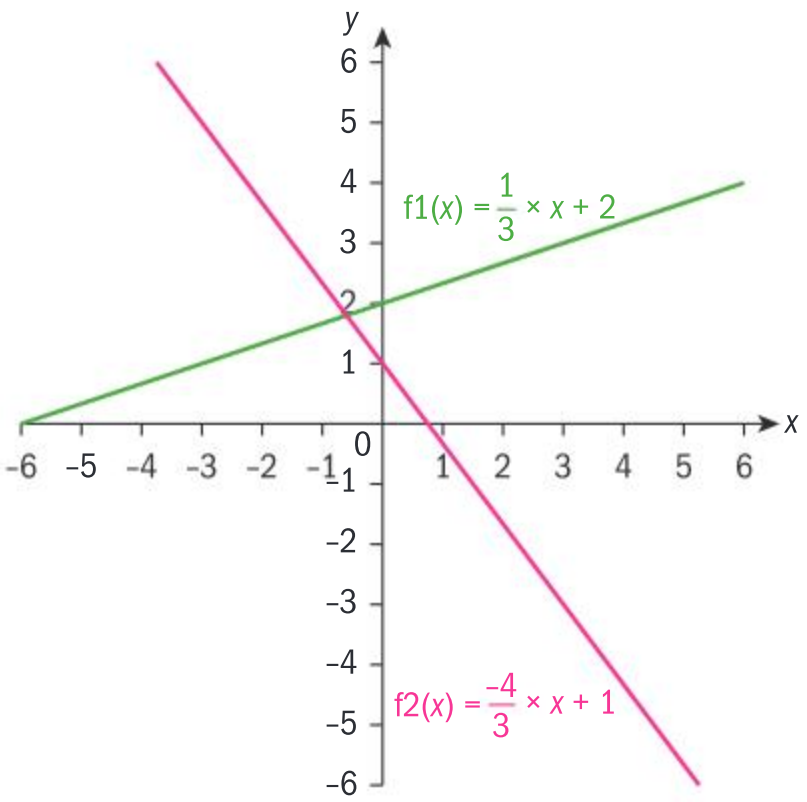
By taking a linear combination of such powers, for example $3 \times x^5 - 2 \times x^2 + 8 \times x - 11 \times 1$, you obtain a polynomial $f(x) = 3x^5 - 2x^2 + 8x - 11$.

Polynomial functions are the functions that you have worked with most frequently so far. Here are some examples of polynomials of different degrees that you have learned about.

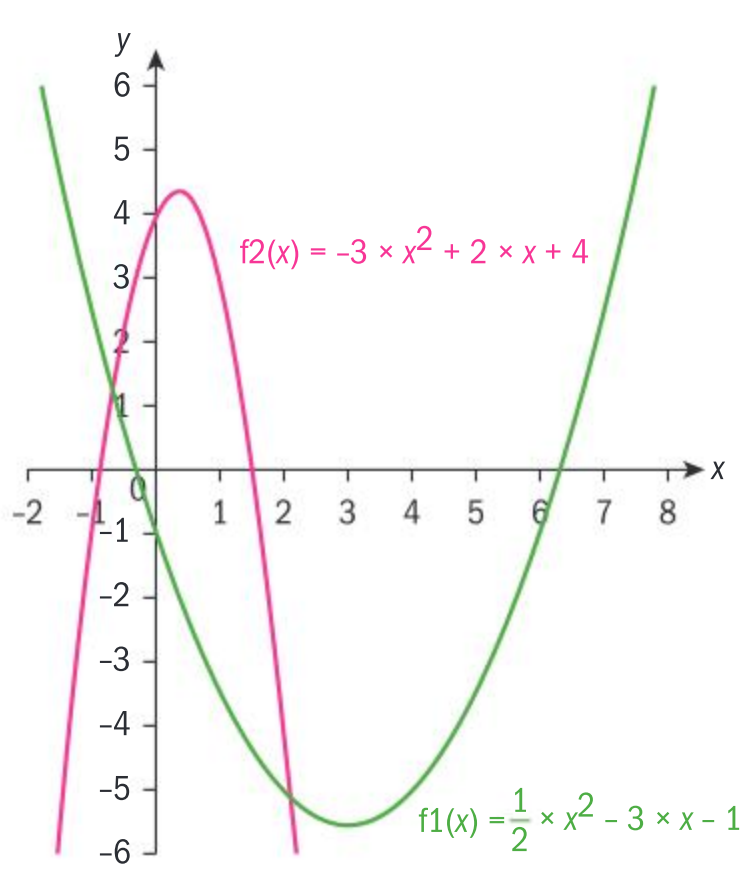
- Zero polynomial: $\theta(x) = 0$. Its graph is a horizontal line along $y = 0$ (the x-axis).
- Constant polynomial: $f(x) = c$, $c \in \mathbb{R}$, $c \neq 0$. Its graph is the horizontal line $y = c$.
- The degree of a constant polynomial is 0.



- Linear polynomial: $f(x) = mx + c$, $m \neq 0$, is also called a polynomial of the first degree. The graph is a straight line. By changing the parameters m and c we change the steepness of the line and the y-axis intercept, respectively.



- Quadratic function: $f(x) = ax^2 + bx + c$, $a \neq 0$, is a polynomial of the second degree. Its graph is a parabola whose axis of symmetry is a vertical line which passes through the vertex of the graph. By changing the parameters a, b and c, we change
  - the shape of the graph (wide or narrow)
  - the concavity (open upwards or downwards)
  - the position of the parabola relative to the coordinate axes.



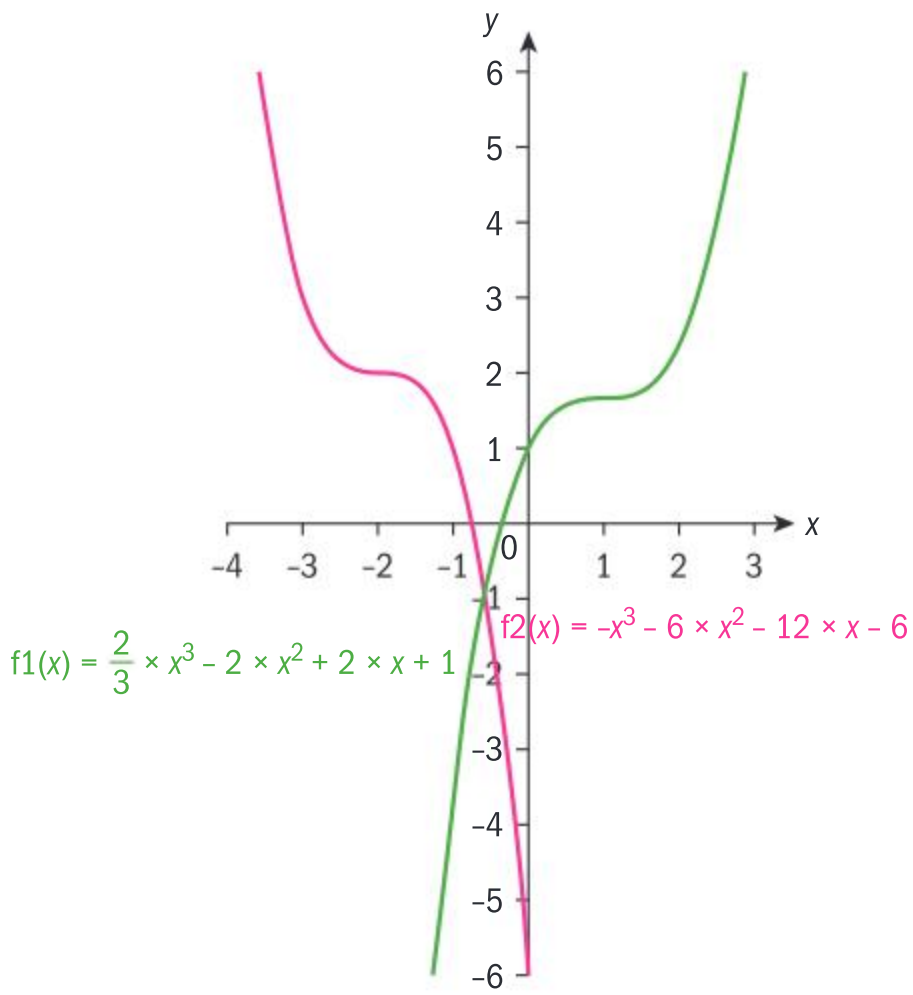
Did you know?

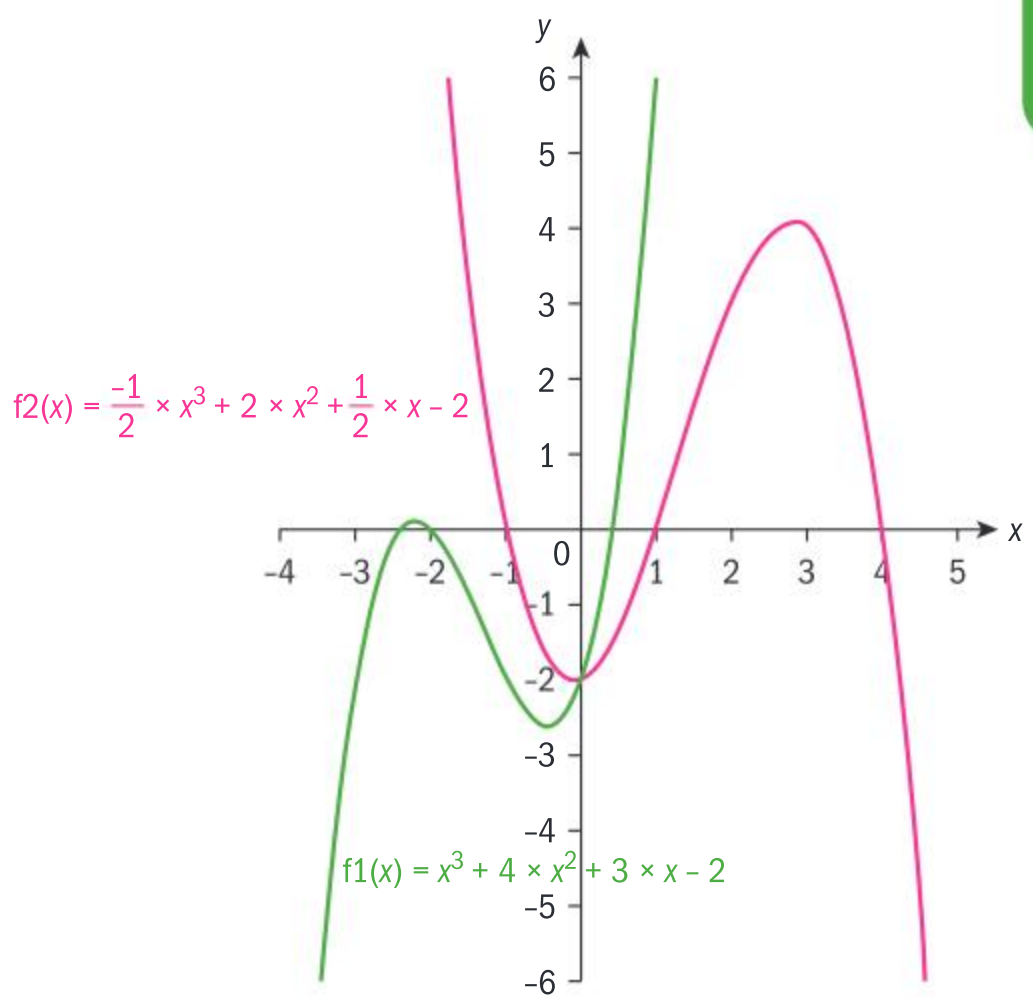
There is still a lot of discussion going on among mathematicians regarding the zero polynomial. Some consider that the zero polynomial is more than just a special case of the constant function. The degree of the zero polynomial is sometimes defined as -1 or even $-\infty$.

HINT

Notice that the notation used for the zero polynomial is $\theta(x) = 0$ so that you can distinguish it from other polynomials. The zero polynomial has an important property, being an additive identity element for the set of polynomials, that is $f(x) + \theta(x) = \theta(x) + f(x) = f(x)$ for all polynomials f.

- Cubic function: $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$, is a polynomial of the third degree. There are two shapes of cubic graphs. One shape looks like a "Flex" shape like those you met in Investigation 10. The second shape is a combination of two U-shapes, one of which is concave up and the other is concave down.





Investigation 11

In Chapter 2, you learned that a quadratic equation $f(x) = ax^2 + bx + c, a \neq 0$, has axis of symmetry $x = -\frac{b}{2a}$ and vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

In this investigation, you will look at how the parameters of a cubic function affect the graph of the function.

Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$.

1. Using graphing software, investigate the effect of changing a. What can you conclude about the effect that a has on the graph?
2. What effect does changing d have on the graph of the function?
3. Investigate the effect of changing the parameter b in the graph of $f(x) = x^3 + bx^2$. How does the shape of the graph change as b changes? Do all graphs of this nature have any common points?
4. Investigate the effect of changing the parameter c in the graph of $f(x) = x^3 + cx$. How does the shape of the graph change as c changes? Do all graphs of this nature have any common points?
5. **Conceptual** How do the parameters in quadratic and cubic functions affect their graphs?

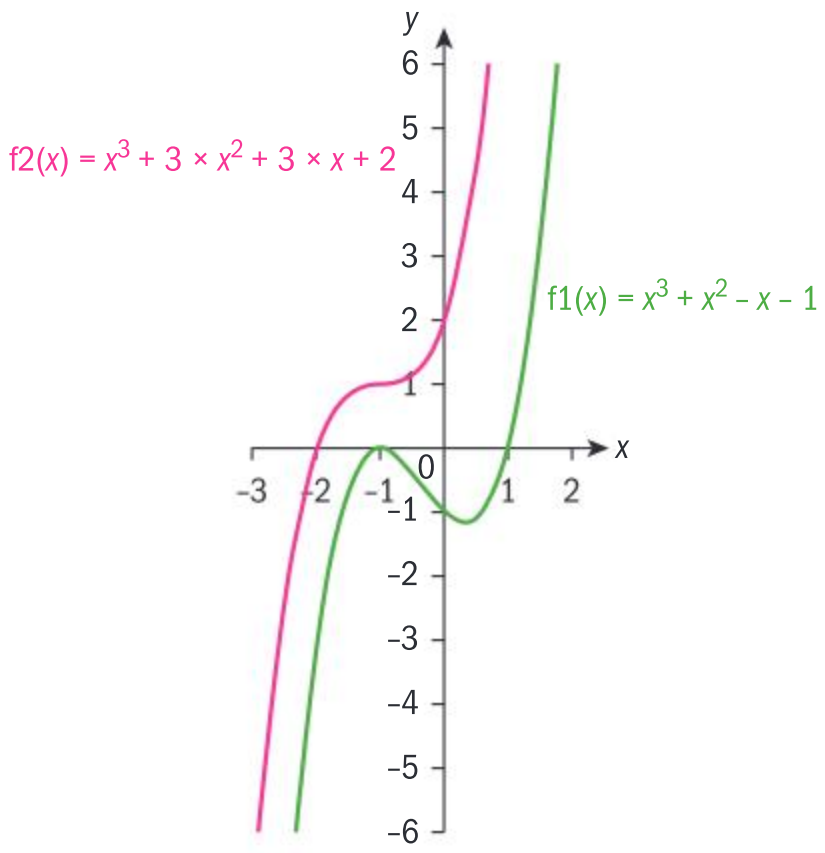
International-mindedness

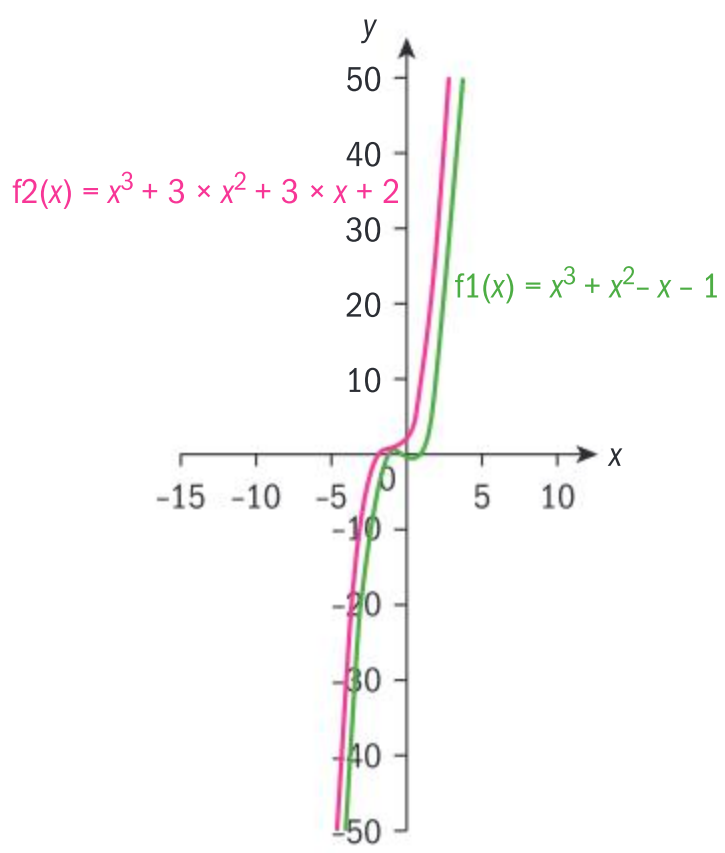
The Sulba Sutras in ancient India and the Bakhshali manuscript contained an algebraic formula for solving quadratic equations.

HINT

With the help of graphing technology, you might want to investigate how the parameters of a fourth-order polynomial affect the shape, symmetry, vertices and intercepts of the graph.

One interesting feature of polynomials of the same degree with the same leading coefficient is that even though locally they might look very different you can find a larger window in which they look very similar.





They behave like a polynomial that has only the leading term, since other terms for extremely large values of x are insignificant to the total value and can be neglected. This is the so-called "end behaviour" property of polynomials.

HINT

Polynomials are continuous functions, which means that you can draw their graphs without lifting your pen from the paper. Notice that you always have to proceed in one direction (usually you draw them from left to right). Their graphs are also smooth therefore they have no sharp points.

The factor and remainder theorems

In order to divide two polynomials you need to remind yourself how to divide two numbers by using long division.

Example 21

Use all the steps in the long division of two numbers to divide $3x^4 + 4x^3 + 2x^2 + 3x - 1$ by $x^2 + x - 2$.

			$3x^2$	$+x$	$+7$
x^2+x-2	$3x^4$	$+4x^3$	$+2x^2$	$+3x$	-1
	$3x^4$	$+3x^3$	$-6x^2$		
		x^3	$+8x^2$	$+3x$	
		x^3	$+x^2$	$-2x$	
			$7x^2$	$+5x$	-1
			$7x^2$	$+7x$	-14
				$-2x$	$+13$

Dividing $3x^4$ by x^2 gives $3x^2$.

Multiply $x^2 + x - 2$ by $3x^2$ and subtract.

Dividing x^3 by x^2 gives x.

Multiply $x^2 + x - 2$ by x and subtract.

Dividing $7x^2$ by x^2 gives 7.

Multiply $x^2 + x - 2$ by 7 and subtract.

Stop when the degree of the remainder is smaller than the degree of the divisor.

You can write:

$$3x^4 + 4x^3 + 2x^2 + 3x - 1 = (x^2 + x - 2)(3x^2 + x + 7) + (-2x + 13)$$



Theorem

For any two polynomials f and g, there are unique polynomials q and r such that $f(x) = g(x) \times q(x) + r(x)$ for all real values of x.

The polynomial q is called the **quotient** and the polynomial r is called the **remainder**. Notice that $\deg(g) > \deg(r)$.

Exercise 3I

1 Use long division to divide polynomial f by polynomial g.

a $f(x) = 2x^3 + 5x^2 - 11x + 4$ and $g(x) = x + 4$

b $f(x) = 3x^5 + x^4 + 6x^3 + 5x^2 + 6$ and $g(x) = x^2 + 2$

c $f(x) = x^6 - x^5 + 2x^4 + x^3 - 5x^2 + 2x - 6$ and $g(x) = x^2 - x + 3$

2 Use long division to find the quotient and remainder when polynomial f is divided by polynomial g.

a $f(x) = 3x^3 - 6x^2 + x - 1$ and $g(x) = x - 1$

b $f(x) = 2x^4 - 3x^3 + 4x^2 + x - 5$ and $g(x) = x^2 + x + 2$

c $f(x) = x^5 + x^4 + 1$ and $g(x) = x^3 + 1$

Polynomial remainder theorem

Given a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $a_n \neq 0$, and a real number p, then the remainder when $f(x)$ is divided by a linear expression $(x - p)$ is $f(p)$.

Proof:

In the unique decomposition of the polynomial $f(x) = (x - p) \times q(x) + r$, where the remainder r is a constant (one degree less than the divisor), we input $x = p \Rightarrow f(p) = \underbrace{(p - p)}_{0} q(p) + r \Rightarrow f(p) = r$.

Factor theorem

A polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n$, $a_n \neq 0$, has a factor $(x - p)$, $p \in \mathbb{R}$, if and only if $f(p) = 0$.

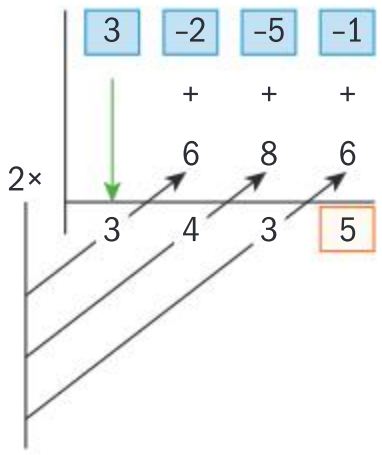
The factor theorem is a direct consequence of the remainder theorem and the proof is left as an exercise for you.

Mathematician William George Horner (1786–1837) formalized an algorithm that simplifies the process of finding the value of a polynomial at a given value of x.

If you want to find the value of $f(x) = 3x^3 - 2x^2 - 5x - 1$ when $x = 2$, select the coefficients of all terms, including missing terms, and organize them in tabular form.

Did you know?

The polynomial remainder theorem is also known as "Little Bézout's theorem" as it was proposed by Étienne Bézout (1730–1837). Bézout was inspired by the work of Euler and decided to become a mathematician. In 1763 he was appointed examiner of the Gardes de la Marine (French Naval Academy) with a special task to compose a textbook for teaching mathematics to the students.



HINT

$$\begin{aligned} f(x) &= ((3 \times x - 2) \times (x - 5)) \times x - 1 \\ f(2) &= ((3 \times 2 - 2)2 - 5) \times 2 - 1 \\ &= ((6 - 2) \times 2 - 5) \times 2 - 1 \\ &= (8 - 5) \times 2 - 1 \\ &= 3 \times 2 - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

Horner's algorithm finds the quotient and remainder when a polynomial $f(x)$ is divided by a linear polynomial $g(x)$. This algorithm is also known as **synthetic division**. Notice that synthetic division only works when you divide a polynomial by a linear polynomial.

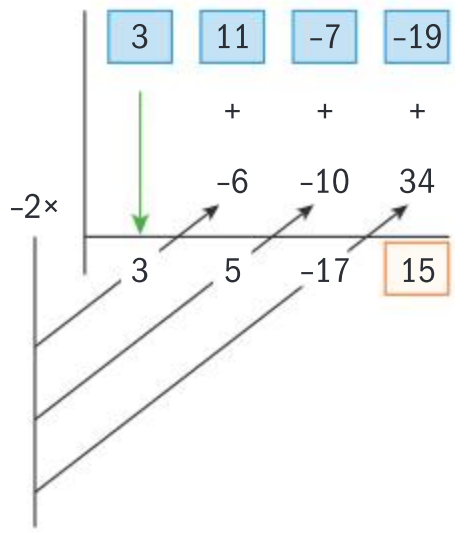
Since synthetic division tells us the quotient when f is divided by a linear factor g, it is an effective and quick method to use when searching for factors of f.

Example 22

Use synthetic division to find the remainder when dividing:

a $f(x) = 3x^3 + 11x^2 - 7x - 19$ by $g(x) = x + 2$, **b** $f(x) = 4x^3 - 3x + 8$ by $g(x) = x - 5$.

a $x + 2 = x - (-2) \Rightarrow r = f(-2)$

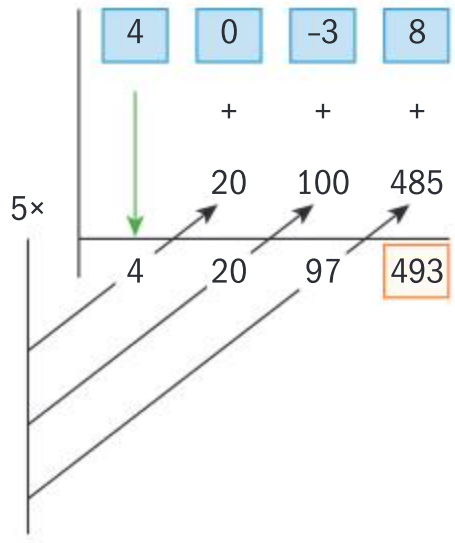


$r = f(-2) = 15$

Use the remainder theorem.

Use synthetic division to find the remainder.

b $x - 5 \Rightarrow r = f(5)$



$r = f(5) = 493$

Use the remainder theorem.

Use synthetic division to find the remainder.



Reflect What do you notice about the quotient polynomial $q(x)$ and the result you obtained from synthetic division in the example above? Can you use the remainder theorem to explain why this is the case?

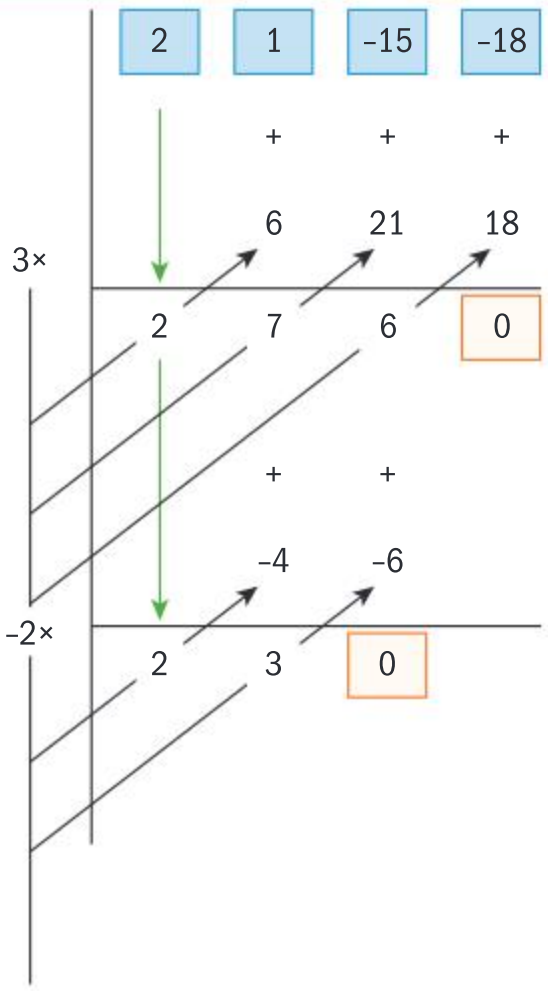
Example 23

The polynomial $f(x) = 2x^3 + x^2 - 15x - 18$ is given.

a Use synthetic division to show that $(x - 3)$ and $(x + 2)$ are factors of the polynomial.

b Hence fully factorize the polynomial.

a



Since the remainder is zero in each case, $(x - 3)$ and $(x + 2)$ are factors of the polynomial.

Use synthetic division twice: first to divide $2x^3 + x^2 - 15x - 18$ by $(x - 3)$, and then to divide the quotient by $(x + 2)$.

Use the factor theorem.

b $2x^3 + x^2 - 15x - 18$

$= (x - 3)(x + 2)(2x + 3)$

In the synthetic division table, you can see the last factor is the second quotient.

Exercise 3J

1 In each case, use synthetic division to find the quotient and the remainder when polynomial f is divided by g.

a $f(x) = x^3 + 2x^2 - 3x + 1$ and $g(x) = x - 2$

b $f(x) = 2x^3 + 3x^2 - 10x - 2$ and $g(x) = x + 3$

c $f(x) = 2x^4 - 6x^3 - 9x^2 + 7x - 11$ and $g(x) = x - 4$

d $f(x) = 3x^5 + 20x^4 + 10x^3 - 13x^2 + 7x - 3$ and $g(x) = x + 6$

2 The polynomial $2x^4 - x^3 - 32x^2 + 31x + 60$ is given.

a Show that $(x + 1)$ and $(x - 3)$ are factors of the polynomial.

b Hence fully factorize the polynomial.

Corollary

Given a polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0, a_k\in\mathbb{R}$, $k=1, 2, \ldots, n, a_n\neq 0$, and real numbers a and $b, a\neq 0$, then the remainder when $f(x)$ is divided by a linear expression $(ax-b)$ is $f\left(\frac{b}{a}\right)$.

The proof is left to you as an exercise.

In order to use synthetic division when dividing by a linear expression $(ax - b)$ we need to modify the algorithm.

$f(x) = (ax - b) \times q(x) + r$

$\Rightarrow f(x) = a\left(x - \frac{b}{a}\right) \times q(x) + r$

$\Rightarrow f(x) = \left(x - \frac{b}{a}\right) \times \left(a \times q(x)\right) + r$

So, when applying Horner's algorithm, notice that all the coefficients in the quotient will have a common factor of a.

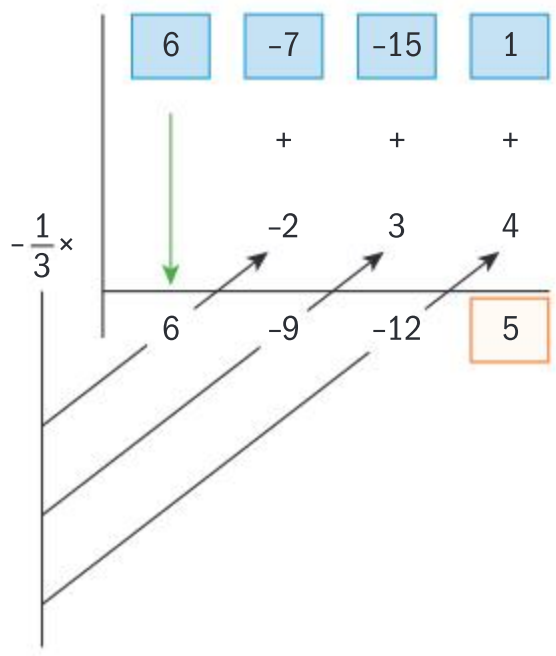
HINT

Use the same technique we used to prove the remainder theorem.

Example 24

Use synthetic division to find the quotient and remainder when dividing $f(x) = 6x^3 - 7x^2 - 15x + 1$ by $g(x) = 3x + 1$.

$g(x) = 3x + 1 = 3\left(x - \left(-\frac{1}{3}\right)\right)$



Rewrite the divisor so that you can use the remainder theorem.

Use Horner's algorithm.

So the quotient is $q(x) = 2x^2 - 3x - 4$ and the remainder is $r(x) = 5$.

$6x^3 - 7x^2 - 15x + 1 = (2x^2 - 3x - 4)(3x + 1) + 5$

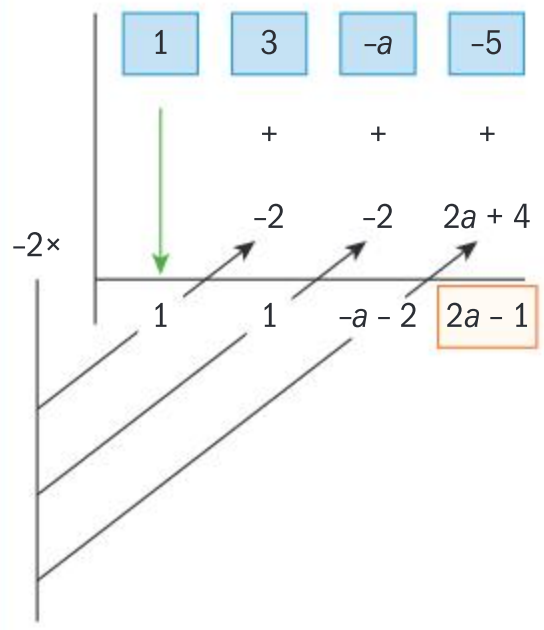
Divide the coefficients of the remainder in the table by 3.



Example 25

When polynomial $f(x) = x^3 + 3x^2 - ax - 5$ is divided by $(x + 2)$ the remainder is 3. Find the value of a.

$x + 2 \Rightarrow r = f(-2)$



$2a - 1 = 3 \Rightarrow a = 2$

Use the remainder theorem.

Use synthetic division to find the remainder in terms of a.

Equate the remainder you obtained in the table with 3, and solve for a.

Example 26

Find the remainder when polynomial $f(x) = 3x^{2019} + 5x^{1019} - 7x + 4$ is divided by $x^2 - 1$.

$$f(x) = (x^2 - 1) \times q(x) + \underbrace{ax + b}_{r(x)}$$

$x = 1 \Rightarrow f(1) = 3 \times 1^{2019} + 5 \times 1^{1019} - 7 \times 1 + 4 = 5$

$x = -1 \Rightarrow f(1) = 3 \times (-1)^{2019} + 5 \times (-1)^{1019} - 7 \times (-1) + 4 = 3$

$$\begin{cases} f(1) = (1^2 - 1) \times q(1) + a \times 1 + b = 5 \\ f(-1) = ((-1)^2 - 1) \times q(-1) + a \times (-1) + b = 3 \end{cases}$$

$$\begin{cases} a + b = 5 \\ -a + b = 3 \end{cases} \Rightarrow \begin{cases} b = 4 \\ a = 1 \end{cases}$$

Therefore the remainder is $r(x) = x + 4$.

Write the unique decomposition of the polynomial and notice that the remainder is a linear polynomial.

Calculate the value of the polynomial at the zeros of the divisor.

Use the unique decomposition of the polynomial and form simultaneous equations.

Solve the simultaneous equations and write the remainder.

Exercise 3K

1 In each case, use synthetic division to find the quotient and the remainder when polynomial f is divided by g.

a $f(x) = 4x^3 - 5x^2 + 13x - 2$ and $g(x) = 4x - 1$

b $f(x) = 6x^3 + 11x^2 + 5x + 4$ and $g(x) = 2x + 3$

c $f(x) = 2x^4 + 3x^3 - 6x^2 + 4x + 3$ and $g(x) = 2x - 1$

d $f(x) = 3x^5 - 5x^4 + x^3 - 2x^2 + 8x + 2$ and $g(x) = 3x + 1$

2 When the polynomial f is divided by the polynomial $g(x) = 2x^2 - 3x + 1$ you obtain the quotient $q(x) = x^2 + 2$ and the remainder $r(x) = x - 3$. Find the polynomial f.

3 $f(x) = 6x^5 + 17x^4 - 20x^3 - 35x^2 + 44x + a$ is divisible by $(x + 2)$. Find the value of a.

4 $f(x) = 2x^4 + 5x^3 - 4x^2 + bx + 1$ is divisible by $(2x + 1)$. Find the value of b.

5 $f(x) = x^4 + 5x^3 + 5x^2 + ax + b$ is divisible by $(x - 1)$ and $(x + 2)$. Find the values of a and b.

6 $f(x) = 6x^5 + 13x^4 - 30x^3 - 45x^2 + ax + b$ is divisible by $(2x - 1)$ and when divided by $(x - 1)$ the remainder is -40. Find the values of a and b.

7 When polynomial f is divided by $(x + 2)$ the remainder is 4, and when divided by $(x - 5)$ the remainder is -3. Find the remainder when polynomial f is divided by $(x^2 - 3x - 10)$.

8 Find the remainder when $f(x) = x^{2019} + x^{2018} + \ldots + x + 1$ is divided by $(x + 1)$.

9 Show that the polynomial $f(x) = (x + 2)^{2n} + (x + 3)^n - 1$ is divisible by $(x^2 + 5x + 6)$ for all $n \in \mathbb{Z}^+$.

10 Given a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, $a_n \neq 0$, and real numbers a and b, $a \neq 0$, show that the remainder when $f(x)$ is divided by the linear expression $(ax - b)$ is $f\left(\frac{b}{a}\right)$.

Developing inquiry skills

Return to the opening problem.

Can you use a polynomial to model the roller coaster?

3.4 The fundamental theorem of algebra

Investigation 12

1 Factorize the following numbers into prime factors. $30, 504, 1155, 35\,200$

2 Fully factorize the following polynomials.
$x^2 + 10x + 25, x^3 - 2x^2 - 15x, x^3 - 3x^2 + 3x - 1, x^4 - 5x^2 + 4$.

3 **Conceptual** What are the similarities and differences between factorizing a number into prime factors, and factorizing a polynomial expression?



The fundamental theorem of algebra is one of the most important theorems in mathematics. It is an algebraic version of the fundamental theorem of arithmetic (which states that every number is either prime or is the product of a unique combination of prime factors.)

The fundamental theorem of algebra refers to the existence of complex zeros of a polynomial.

The fundamental theorem of algebra (FTA)

A polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$, with real or complex coefficients has at least one zero. There is an $\omega \in \mathbb{C}$ such that $f(\omega) = 0$.

Gauss proved this theorem, but the proof goes beyond the level of this textbook.

There are a lot of theorems, lemmas and corollaries that are derived from the fundamental theorem of algebra that can help with algebraic manipulation of equations and polynomial functions.

Lemma

Each polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$, with real coefficients can be written in a factor form, $f(x) = a_n(x - \omega_1)(x - \omega_2)\ldots(x - \omega_n)$, such that $\omega_k \in \mathbb{C}$, $k = 1, \ldots, n$.

Example 27

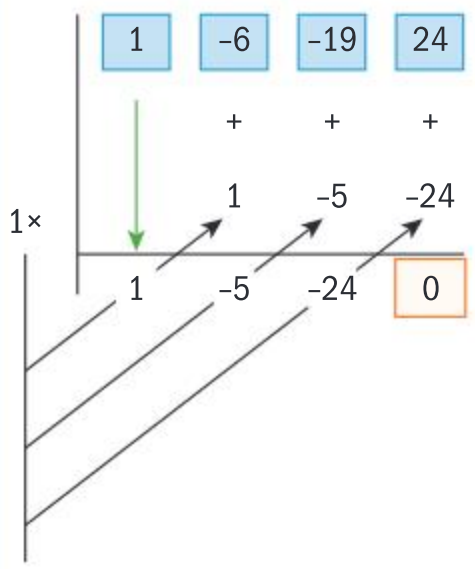
Express the polynomial $f(x) = x^4 - 6x^3 - 19x^2 + 24x$ as a product of linear factors, and check your answer by using your calculator.

$f(x) = x^4 - 6x^3 - 19x^2 + 24x$

$= x(x^3 - 6x^2 - 19x + 24)$

Use the lemma to factorize the polynomial. All the terms have the common factor x, so you can factorize this first.

Notice that the sum of the coefficients of the remaining cubic expression is 0, therefore apply synthetic division for $x = 1$ to factorize it further. $(x - 1)$ is a factor since the remainder is 0.



$x^2 - 5x - 24 = x^2 \underbrace{-8x + 3x}_{-5x} - 24$

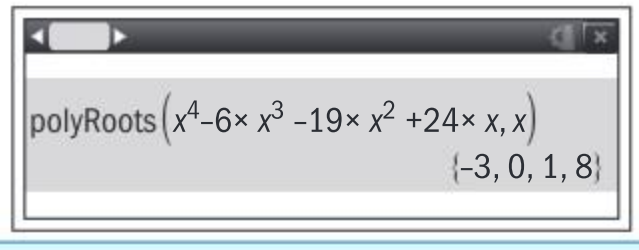
Factorize the remaining quadratic expression by splitting the middle term.

Continued on next page

$= x(x-8) + 3(x-8) = (x-8)(x+3)$

$f(x) = x(x-1)(x+3)(x-8)$

Checking:



Use all the factors in the factor form.

Make sure you know how to use your calculator to check the zeros.

Notice that a calculator gives zeros and you need to use the FTA to write the polynomial in the factor form.

If a certain factor appears more than once we say that the factor has **multiplicity**. So if we have a polynomial f of nth degree with less than n different zeros, then the sum of their multiplicities will add up to n.

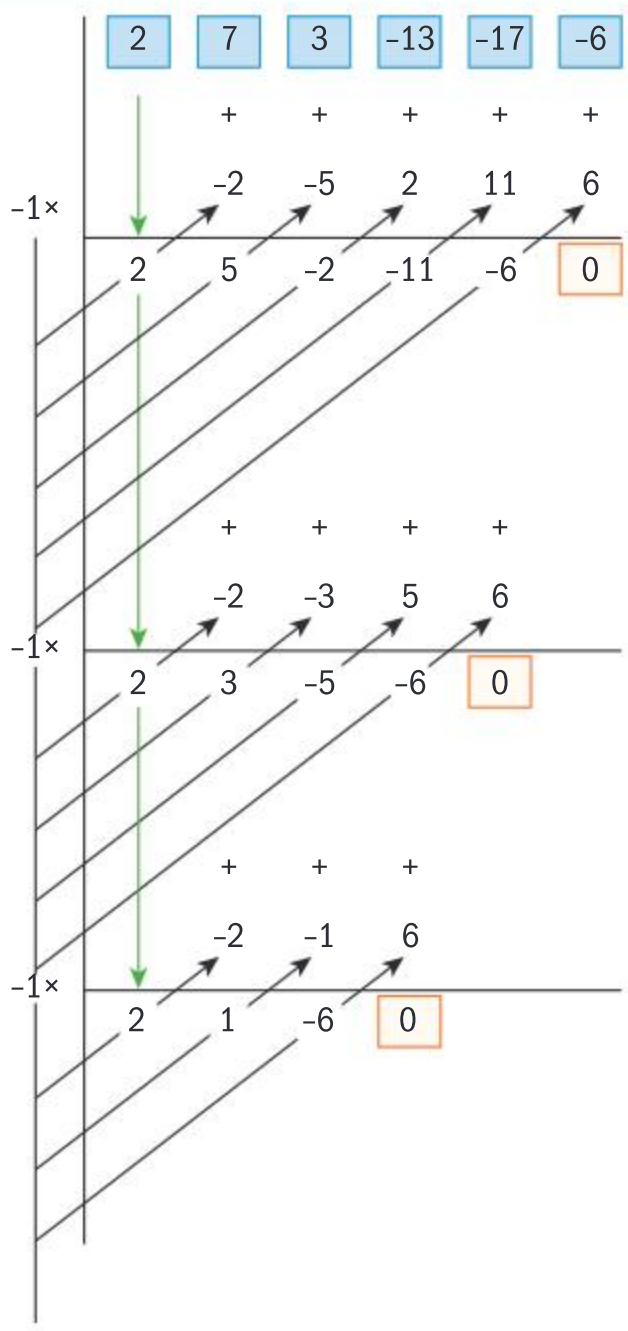
$$f(x) = a_n(x-\omega_1)^{p_1}(x-\omega_2)^{p_2}\ldots(x-\omega_k)^{p_k},\ k<n,\ \sum_{i=1}^{k} p_k = n$$

Example 28

-1 is a zero of the polynomial $f(x) = 2x^5 + 7x^4 + 3x^3 - 13x^2 - 17x - 6$ with multiplicity 3.

Express $f(x)$ as the product of linear factors.

Check your answers by using a calculator.



Successively apply synthetic division with respect to the multiplicity of the given zero, in this case three times.

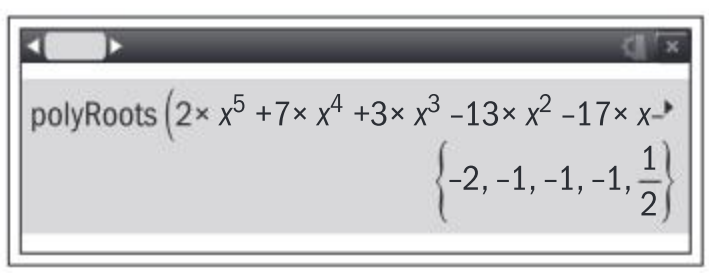


$2x^2 + x - 6 = 2x^2 \underbrace{-3x + 4x}_{x} - 6$

$= x(2x - 3) + 2(2x - 3) = (2x - 3)(x + 2)$

$f(x) = (x + 1)^3(2x - 3)(x + 2)$

Checking:



Factorize the remaining quadratic expression by splitting the middle term.

Use all the factors in the factor form.

Use your calculator to check the zeros.

Notice that the last zero is in fraction form, so you need to multiply it by 2 to obtain the linear factor:

$2\left(x - \frac{3}{2}\right) = (2x - 3)$

Investigation 13

1 Use the remainder theorem and synthetic division to show that the complex number z is a zero of the polynomial f.

a $z = \mathrm{i}, f(x) = 2x^3 - 3x^2 + 2x - 3$

b $z = -4\mathrm{i}, f(x) = 2x^3 - x^2 + 32x - 16$

c $z = -2 + 3\mathrm{i}, f(x) = 3x^3 + 10x^2 + 31x - 26$

d $z = 1 + \mathrm{i}, f(x) = x^4 + x^3 - 2x^2 + 2x + 4$

e $z = -1 + 3\mathrm{i}, f(x) = x^5 + x^4 + 9x^3 - 9x^2 + 8x - 10$

2 Use the remainder theorem and synthetic division to find the values of the polynomial f at z^* for all the complex numbers given in question **1**.

3 **Conceptual** If z is a zero of a polynomial f, what can you say about the complex conjugate z^*?

4 **Factual** By considering the fundamental theorem of algebra and your answer to question **3**, what can you say about the zeros of an odd-degree polynomial?

Did you know?

Due to the imperfection of the calculator's algorithm when finding zeros, you obtain approximations of the multiple zeros, without their multiplicity. This will occur even more times when complex zeros occur.

TOK

Aliens might not be able to speak an Earth language but would they still describe the equation of a straight line in similar terms?

Is mathematics a formal language?

Your findings from Investigation 13 can be summarized by the following theorem.

Conjugate root theorem
Given a polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,\ a_k \in \mathbb{R},\ k = 1, 2, \ldots, n,\ a_n \neq 0,$

that has a complex zero z, then its conjugate z^* is also a zero of the polynomial f.

Example 29

Given that $1 + 3i$ is a complex zero of the polynomial $f(x) = x^3 - 5x^2 + 16x - 30$, find all the other zeros of f.

Check your answers using a calculator.

Method 1

Use the conjugate zero theorem.

$x_1 = 1 + 3i \Rightarrow x_2 = 1 - 3i$ is a zero of f

$(x - (1 + 3i))(x - (1 - 3i))$

$= x^2 - x - 3ix - x + 3ix + 1^2 + 3^2 = x^2 - 2x + 10$

Expand and find the quadratic factor.

Use long division to find the remaining factor.

$$
\begin{array}{r|llll}
 & x & -3 & & \\
\hline
x^2-2x+10 & x^3 & -5x^2 & 16x & -30 \\
 & -x^3 & -2x^2 & +10x & \\
\hline
 & 0 & -3x^2 & 6x & -30 \\
 & & -3x^2 & +6x & -30 \\
\hline
 & & 0 & 0 & 0
\end{array}
$$

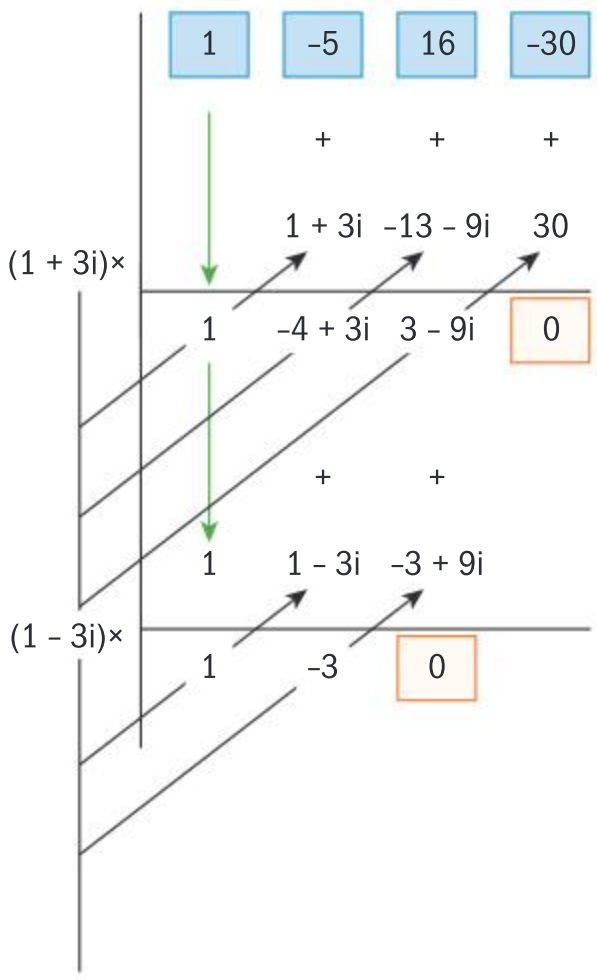
$f(x) = (x - (1 + 3i))(x - (1 - 3i))(x - 3)$

$x - 3 = 0 \Rightarrow x = 3$

Fully factorize the polynomial and find the remaining zero.

Method 2

Use synthetic division for complex numbers twice.

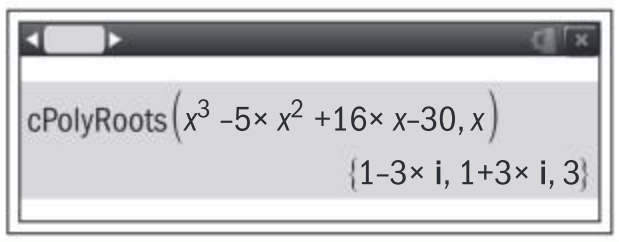


$f(x) = (x - (1 + 3i))(x - (1 - 3i))(x - 3)$

$x - 3 = 0 \Rightarrow x = 3$

Fully factorize the polynomial and find the remaining zero.

Checking:



Make sure that you know how to use your calculator to check your answers. Notice that you need to use complex roots of polynomials.



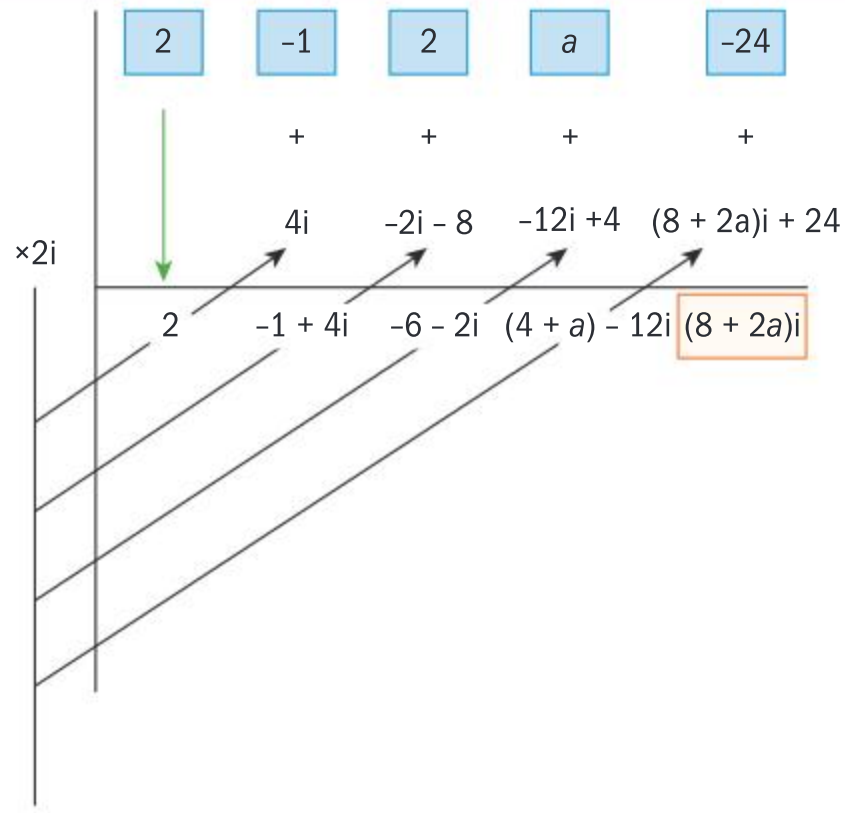
Example 30

The polynomial $f(x) = x^4 - x^3 + 2x^2 + ax - 24, a \in \mathbb{R}$, is given.

a Find the value of a given that 2i is a complex zero of f.

b Hence find all the remaining zeros of f, and check your answers using a calculator.

a



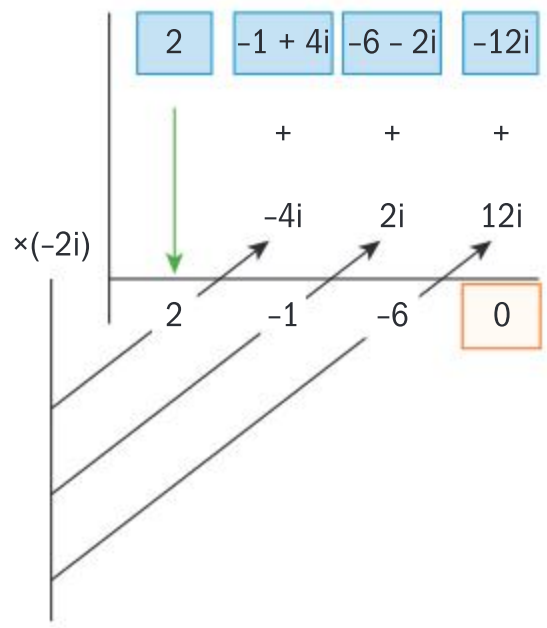
Apply synthetic division for the complex number 2i.

$f(2\text{i}) = (8 + 2a)\text{i} = 0 \Rightarrow a = -4$

Use the remainder theorem.

b $x_1 = 2\text{i} \Rightarrow x_2 = -2\text{i}$ is a root of f

Use the conjugate zero theorem.



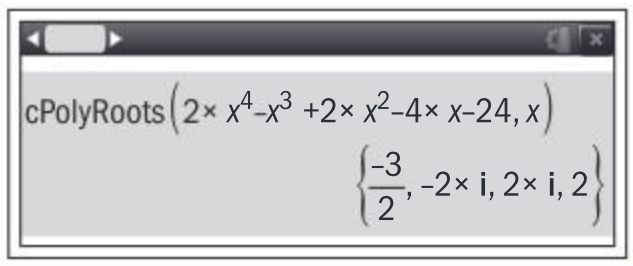
Use $a = -4$ and apply synthetic division a second time.

$q(x) = 2x^2 - x - 6 = 0 \Rightarrow (2x + 3)(x - 2) = 0$

$\Rightarrow 2x + 3 = 0$ or $x - 2 = 0 \Rightarrow x = -\frac{3}{2}$ or $x = 2$

Identify the quotient and find its zeros.

Checking:



Use your calculator to check your answers.

Exercise 3L

1 Find a polynomial of the smallest degree with integer coefficients whose zeros are

a 0, 2 and 7

b $-3, -2, 1$ and 3

c $-1, -\frac{1}{2}$, 2 and 5.

2 Find a polynomial of the smallest degree with integer coefficients whose zeros include

a $\sqrt{2}$ and 2

b $-1, \frac{1}{2}$ and $\sqrt[3]{3}$

c $1-\sqrt{3}$ and $\sqrt[3]{2}$

3 Factorize the following polynomials and check your answers using a calculator.

a $f(x) = x^3 - 3x^2 + 4x - 2$

b $f(x) = 3x^3 - x^2 + 2x + 6$

c $f(x) = 2x^4 - 5x^3 + 11x^2 - 3x - 5$

4 In each case, k is a zero with multiplicity n of the polynomial f.

Factorize f fully and check your answers using a calculator.

a $k = -2, n = 2, f(x) = 3x^3 + 7x^2 - 8x - 20$

b $k = \frac{2}{3}, n = 2, f(x) = 9x^3 + 24x^2 - 44x + 16$

c $k = 1, n = 2, f(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$

d $k = -\frac{1}{2}, n = 3, f(x) = 8x^4 + 20x^3 + 18x^2 + 7x + 1$

e $k = 1, n = 4, f(x) = 5x^5 - 13x^4 + 2x^3 + 22x^2 - 23x + 7$

5 In each case, f is a polynomial with complex zero z. Find the remaining zeros.

a $f(z) = z^3 - 2z^2 + 4z - 8, \ z = 2\text{i}$

b $f(z) = 2z^3 - 13z^2 + 32z - 13, \ z = 3 - 2\text{i}$

c $f(z) = 3z^3 + 4z^2 - 4z + 7, \ z = \frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}$

d $f(z) = z^4 - 2z^3 + 6z^2 - 2z + 5, \ z = -\text{i}$

e $f(z) = z^4 + 5z^3 + 11z^2 + 13z + 10, \ z = -2 - \text{i}$

f $f(z) = 3z^4 + 2z^3 + 19z^2 + 12z + 6, \ z = -\frac{1}{3} + \frac{\sqrt{2}}{3}\text{i}$

6 Given that z is a zero of the polynomial f, find the missing coefficients. Hence find all the remaining zeros of f and check your answers using a calculator.

a $z = 2, f(x) = x^3 + ax^2 + x - 2, \ a \in \mathbb{R}$

b $z = -5, f(x) = 2x^3 + 10x^2 + ax + 15, \ a \in \mathbb{R}$

c $z = \sqrt{5}\,\text{i}, f(x) = x^4 - 2x^3 + ax^2 + bx + 85, \ a, b \in \mathbb{R}$

d $z = 1 - 2\text{i}, f(x) = 3x^4 - 7x^3 + 18x^2 + ax + b, \ a, b \in \mathbb{R}$

Sum and product of polynomial roots

Just as you have already shown for quadratic polynomials, François Viète developed formulas that connect the zeros and the coefficients of a polynomial. Viète was the first to investigate this connection but he did it just for positive real zeros, whilst Albert Girard was the first to extend it to complex zeros.

Investigation 14

1 Use your calculator to find the zeros of the following polynomials.

$f_1(x) = x^3 + x^2 - 4x - 4, f_2(x) = x^3 - 6x^2 + 11x - 6,$

$f_3(x) = x^3 - 4x^2 - 7x + 10, f_4(x) = x^3 + 27x^2 + 71x - 1155.$

International-mindedness

François Viète (1540–1603) was a French amateur mathematician and astronomer, whilst Albert Girard was a French mathematician and musician.



2 Find the sums and the products of all the zeros for each polynomial in question **1**.

3 **Factual** How do the sum and product of the zeros you found in question **2** relate to the coefficients of each polynomial?

4 Use your calculator to find the zeros of the following polynomials.

$f_5(x) = 2x^3 - 11x^2 + 17x - 6, f_6(x) = 3x^3 - 16x^2 + 3x + 10,$

$f_7(x) = 15x^3 + 19x^2 - 4, f_8(x) = 14x^3 + 291x^2 + 1402x - 1155.$

5 Find the sums and the products of all the zeros for each polynomial in question **4**.

6 **Conceptual** How do the sum and product of the zeros of a cubic relate to their coefficients?

Your results from Investigation 14 form part of the following key point.

Given a cubic equation $ax^3 + bx^2 + cx + d = 0, a, b, c, d \in \mathbb{R}, a \neq 0$, with x_1, x_2 and x_3 as solutions, then

$$\begin{cases} x_1 + x_2 + x_3 = -\dfrac{b}{a} \\ x_1 \times x_2 + x_1 \times x_3 + x_2 \times x_3 = \dfrac{c}{a} \\ x_1 \times x_2 \times x_3 = -\dfrac{d}{a} \end{cases}$$

Example 31

The roots of a cubic equation $3x^3 + 5x^2 + 2x - 4 = 0$ are x_1, x_2 and x_3.

Without solving the equation, find

a $x_1 + x_2 + x_3$ **b** $x_1 \times x_2 \times x_3$ **c** $\dfrac{1}{x_1 x_2} + \dfrac{1}{x_1 x_3} + \dfrac{1}{x_2 x_3}$.

$a = 3, b = 5, c = 2, d = -4$	Identify the coefficients of the cubic polynomial.
a $x_1 + x_2 + x_3 = -\dfrac{5}{3}$	Use the theorem for the sum.
b $x_1 \times x_2 \times x_3 = -\dfrac{-4}{3} = \dfrac{4}{3}$	Use the theorem for the product.

Continued on next page

c $\frac{1}{x_1x_2}+\frac{1}{x_1x_3}+\frac{1}{x_2x_3}=\frac{x_3+x_2+x_1}{x_1x_2x_3}=\frac{-\frac{5}{3}}{\frac{4}{3}}=-\frac{5}{4}$

Rewrite the expression in terms of the sum and product and then use the results from the previous parts.

Make sure you know how to use a calculator to check your results.

Use the complex roots finder to obtain all the possible zeros. Use the features for sum and product to find the results.

Notice that the results obtained in decimal form are the same as the results in fraction form to a certain degree of accuracy.

Investigation 15

1 Use your calculator to find the zeros of the following polynomials.

$f_1(x)=x^4+x^3-7x^2-x+6, f_2(x)=x^4+4x^3-39x^2-86x+280,$

$f_3(x)=6x^4+7x^3-13x^2-4x+4, f_4(x)=35x^4+381x^3-73x^2-309x+110,$

$f_5(x)=4x^4-4x^3+65x^2-64x+16, f_6(x)=6x^4-11x^3+117x^2+397x-145.$

2 Find the sums and the products of all the zeros for each polynomial in question **1**.

3 **Conceptual** How do the sum and product of the zeros of a fourth-degree polynomial relate to its coefficients?

Given a polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ with real or complex coefficients $(a_n\neq 0)$ and zeros $x_1, x_2, \ldots, x_n$ of f, then

$$x_1+x_2+x_3+\ldots+x_n=-\frac{a_{n-1}}{a_n}$$

and $$x_1x_2x_3\ldots x_n=(-1)^n\frac{a_0}{a_n}$$

The proof comes as a direct consequence of the factor theorem, as you saw when proving the equivalent result for a cubic polynomial. Since every polynomial can be written in the form $f(x)=a_n(x-x_1)(x-x_2)\ldots(x-x_n)$, by expanding this expression we obtain the given formulas.



Example 32

Find the sum and the product of the zeros of the following polynomials.

a $f(x) = x^4 + x^3 - 2x^2 + 3x - 2$

b $f(x) = 2x^5 - 5x^4 + 15x^3 - 511x^2 - 4x + 8$

c $f(x) = -5x^{2019} + 7x^{1457} - 4x^{250} - 15x^{47} + 21$

a $n = 4,\ a_4 = 1,\ a_3 = 1,\ a_0 = -2$

$$x_1 + x_2 + x_3 + x_4 = -\frac{a_3}{a_4} = -\frac{1}{1} = -1$$

$$x_1 x_2 x_3 x_4 = (-1)^4 \frac{a_0}{a_4} = \frac{-2}{1} = -2$$

Use the formulas for sum and product from the key point.

Notice that for those formulas you need to use only the three coefficients.

b $n = 5,\ a_5 = 2,\ a_4 = -5,\ a_0 = 8$

$$\sum_{k=1}^{5} x_k = -\frac{a_4}{a_5} = -\frac{-5}{2} = \frac{5}{2}$$

$$\prod_{k=1}^{5} x_k = (-1)^5 \frac{a_0}{a_5} = -\frac{8}{2} = -4$$

c $n = 2019,\ a_{2019} = -5,\ a_{2018} = 0,\ a_0 = 21$

$$\sum_{k=1}^{2019} x_k = -\frac{a_{2018}}{a_{2019}} = -\frac{0}{-5} = 0$$

$$\prod_{k=1}^{2019} x_k = (-1)^{2019} \frac{a_0}{a_{2019}} = -\frac{21}{-5} = \frac{21}{5}$$

Use your calculator to check the results.

Notice that sometimes, due to the accuracy of the calculator's algorithm, the results are not exact (the complex part is almost 0).

In this example the calculator cannot even find the zeros due to the high order, so it is helpful to have an algebraic technique to find their sum and product.

Exercise 3M

1 Find the sum and product of the zeros of these polynomials.

a $f(x) = x^4 - 3x^3 + 2x^2 - 5x + 4$

b $f(x) = 2x^6 - 6x^5 + 11x^4 + 13x^3 - 5x$

c $f(x) = 23x^{17} - \frac{5}{11}x^9 + \sqrt{13} \times x^4 - \frac{\pi}{3}x - 46$

d $f(x) = 3x^{2020} - 4x^{2019} + 17x^{573} + 115x - 8$

2 The cubic equation $4x^3 - 2x^2 + x - 7 = 10$ has roots x_1, x_2 and x_3. Without solving the equation, find

a $x_1 + x_2 + x_3$

b $x_1 \times x_2 \times x_3$

c $10x_1 + 10x_2 + 10x_3$

d $\frac{3}{x_1 \times x_2} + \frac{3}{x_1 \times x_3} + \frac{3}{x_2 \times x_3}$

Check your results using a calculator.

3 The roots of the quartic equation
$6x^4 + 2x^3 - 11x^2 + 13x - 3 = 0$
are x^1, x^2, x^3 and x^4. Without solving the equation, find

a $x_1 + x_2 + x_3 + x_4$

b $x_1 \times x_2 \times x_3 \times x_4$

c $3x_1 + 3x_2 + 3x_3 + 3x_4$

d $\frac{6}{x_1 \times x_2 \times x_3} + \frac{6}{x_1 \times x_2 \times x_4} + \frac{6}{x_1 \times x_3 \times x_4} + \frac{6}{x_2 \times x_3 \times x_4}$

Check your results using a calculator.



3.5 Solving equations and inequalities

Since you have already extensively covered solving quadratic equations in Section 3.1, in this section you are mainly going to be focused on solving polynomial equations of degree 3 or higher.

Historically, a group of Italian mathematicians, dal Ferro, Tartaglia, and Cardano (13th–14th century), developed a formula for solving a general cubic equation.

That formula is very complicated and very difficult to use, so you are going to be studying rational solutions that can be found by alternative, simpler methods.

The most common method that you are going to use for solving polynomial equations is factorization. In order to be able to factorize the polynomial you are going to use theorems that are valid for polynomials of all degrees.

Before you start on factorization you are going to look at how many real zeros you may expect for a given polynomial.

Did you know?

Babylonians were able to solve some cases of cubic equations (2000–1600 BC). They had tables with perfect squares and perfect cubes and their sums. By using those tables, they were solving equations of the form $ax^3 + bx = c$.

Solving polynomial equations

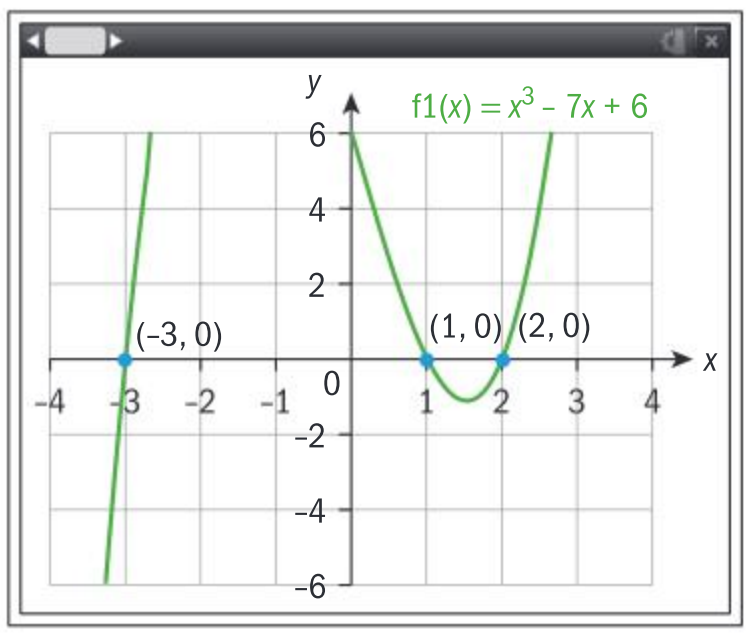
In this section you are going to solve polynomial equations by using algebraic methods and then use a calculator to verify the solutions.

Example 33

Solve the equation $x^3 - 7x + 6 = 0$ by using algebra.

$x^3 - 7x + 6 = \underbrace{x^3 - x}\,\underbrace{-6x + 6}$	Split the linear term so that you can factorize in pairs.
$= x(x^2 - 1) - 6(x - 1) = x(x - 1)(x + 1) - 6(x - 1)$	Factorize the difference of two squares and then factorize the expression.
$= (x - 1)(x(x + 1) - 6)$	Factorize the quadratic factor by inspection.
$= (x - 1)(x^2 + x - 6) = (x - 1)(x - 2)(x + 3)$	
$\Rightarrow (x - 1)(x - 2)(x + 3) = 0$	Use zero product theorem.
$\Rightarrow x_1 = 1, x_2 = 2, x_3 = -3$	
	Make sure you know how to check the result by finding the zeros of the polynomial using a calculator.

On a calculator you can also use the graphical method for finding zeros of the polynomial.



Apart from quadratics, you haven't drawn many graphs of higher-order polynomials. You are going to graph more polynomials once calculus is introduced.

Exercise 3N

1 Solve the following equations. Give all the real solutions, and check your answers using a calculator.

a $x^3 + x^2 - 4x - 4 = 0$

b $x^3 + 2x^2 - 9x - 18 = 0$

c $x^3 - 3x^2 + 3x - 2 = 0$

d $x^4 + x^3 - 3x^2 - x + 2 = 0$

2 The equation $x^3 + x^2 + ax - 4 = 0$ has one root equal to -2.

a Find the value of a.

b Find the remaining two roots.

3 The equation $2x^3 + ax^2 + bx + 9 = 0$ has one repeated root equal to 3.

a Find the values of a and b.

b Find the remaining root.

4 $f(x) = x^3 + ax^2 + bx + c, a, b, c \in \mathbb{R}$.
Two zeros of this polynomial are opposite numbers.

a Show that $ab = c$.

b Find the third zero.

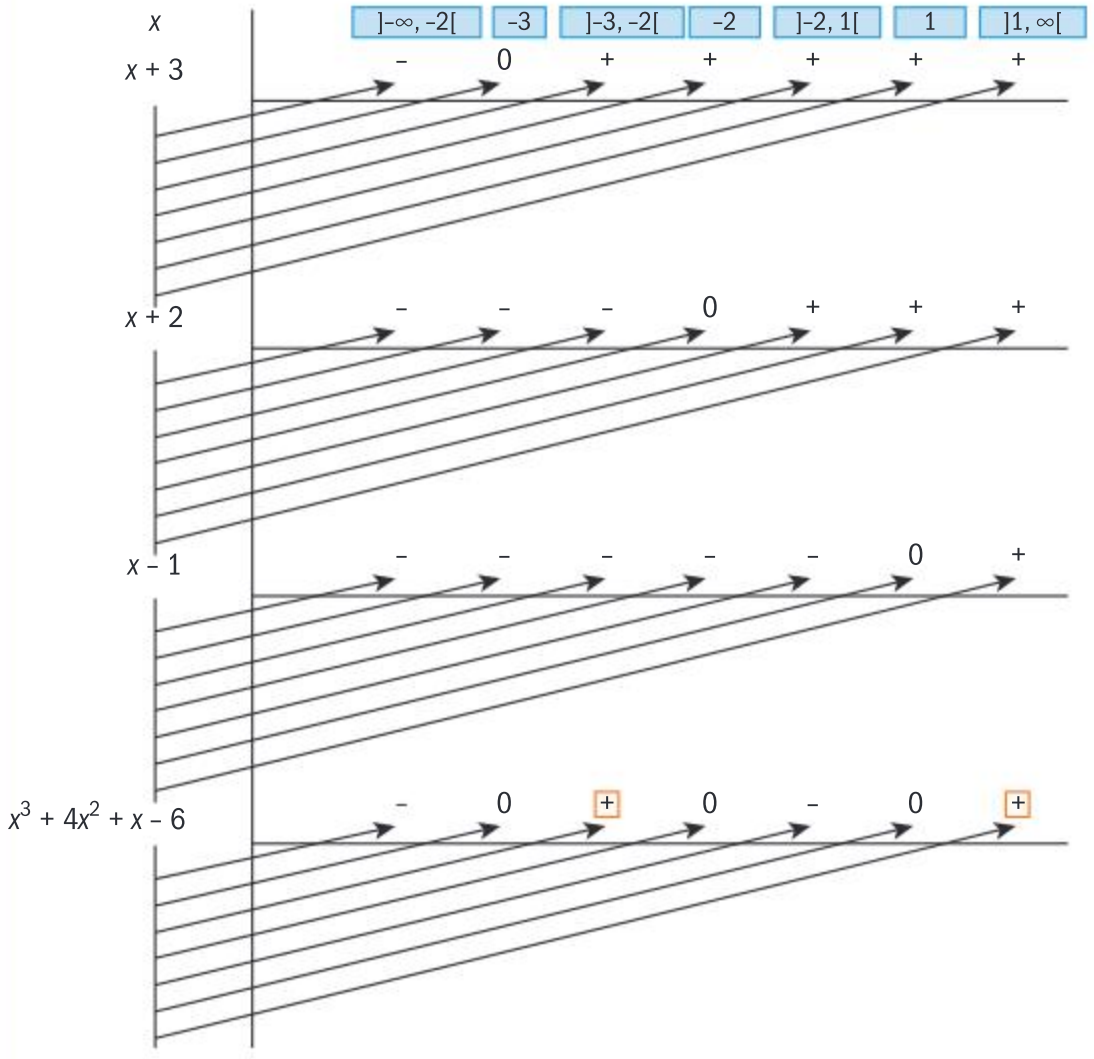
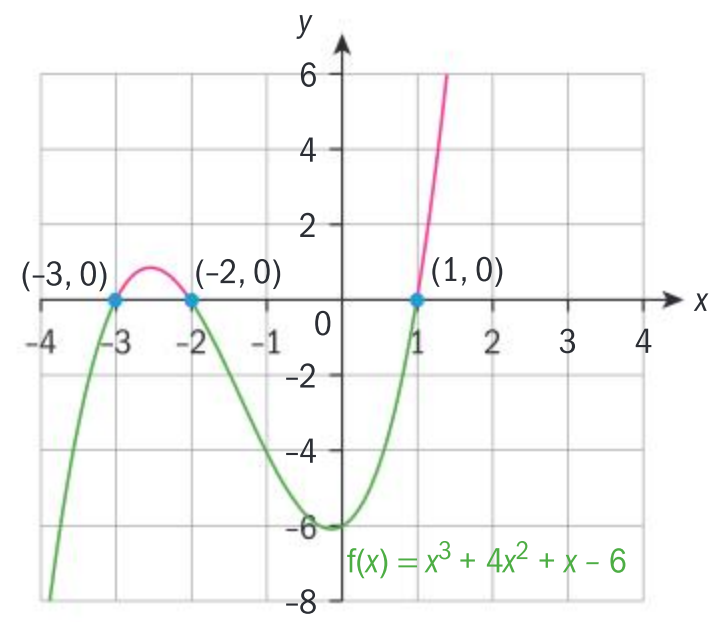
Solving polynomial inequalities

When solving polynomial inequalities by an algebraic method, you need to fully factorize the polynomial and then investigate the signs of the factors on the whole set of real numbers by constructing a sign table. You should identify for which values of x the polynomial is either positive or negative, and use this to solve the inequality.

When solving polynomial inequalities by a graphical method, use a calculator to graph the polynomial and again identify for which values of x the polynomial is either positive or negative, and use this to solve the inequality.

Example 34

Use an algebraic method to solve the inequality $x^3 + 4x^2 + x - 6 > 0$ and check your answer using a calculator.

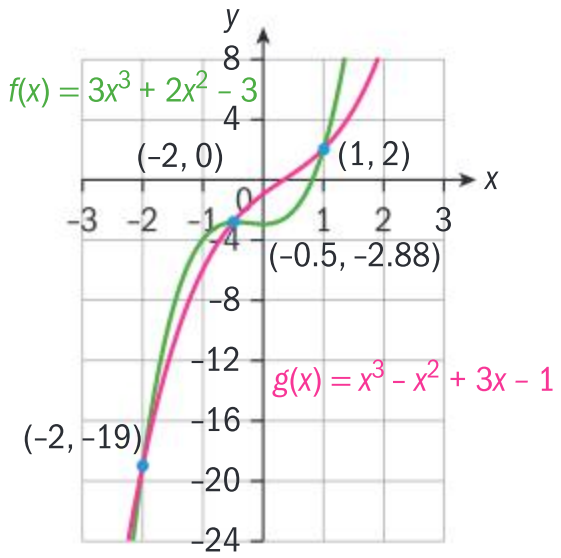
Factorization: $x^3 + 4x^2 + x - 6 = x^3 + 4x^2 + 4x - 3x - 6$ $= x(x^2 + 4x + 4) - 3(x + 2) = x(x + 2)^2 - 3(x + 2)$ $= (x + 2)(x(x + 2) - 3) = (x + 2)(x^2 + 2x - 3)$ $= (x + 2)(x - 1)(x + 3)$	Factorize the expression by splitting the linear term. Use the distributive property to factorize the expression. Factorize the remaining quadratic factor.
	Construct the sign table and investigate the sign.
$x \in]-3, -2[\cup]4, \infty[$	Identify the intervals where the sign is positive.
	Draw the graph of the polynomial and find all the zeros. Identify the values of x for which the graph is above the x-axis.

Notice that when the inequality is not strict, you need to include the boundaries in the solution.

Example 35

Given the polynomials $f(x) = 3x^3 + 2x^2 - 3$ and $g(x) = x^3 - x^2 + 3x - 1$, find all the values of x such that $f(x) \geq g(x)$ by using a graphical method on a calculator.

Method 1



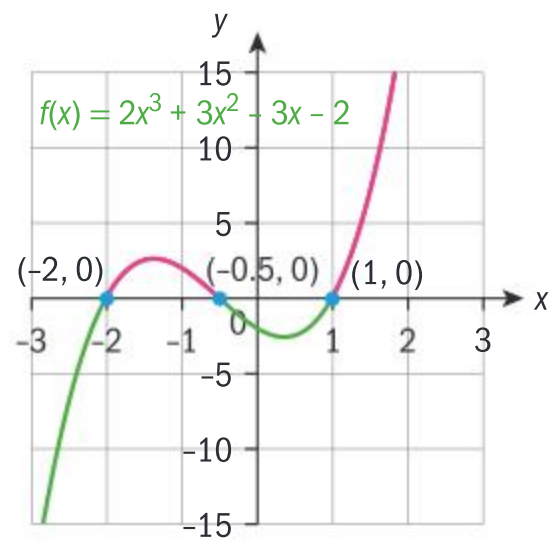
$x \in [-2, -0.5] \cup [1, \infty[$

Graph both polynomials and find the points of intersection.

Method 2

$3x^3 + 2x^2 - 3 \geq x^3 - x^2 + 3x - 1$

$2x^3 + 3x^2 - 3x - 2 \geq 0$



$x \in [-2, -0.5] \cup [1, \infty[$

Identify the intervals on the x-axis where the graph of f is above the graph of g.

Rewrite the inequality as one polynomial and find its zeros. Then find the intervals where the polynomial is above or on the x-axis.

HINT

In the case of Method 1 you may have needed to consider changing the scale on your GDC to find all the points of intersection between the graphs that might not be visible in the original window. Also, it can be difficult to read from the graph which function is the upper one and which the lower one. This is even more difficult on calculators with lower resolution.

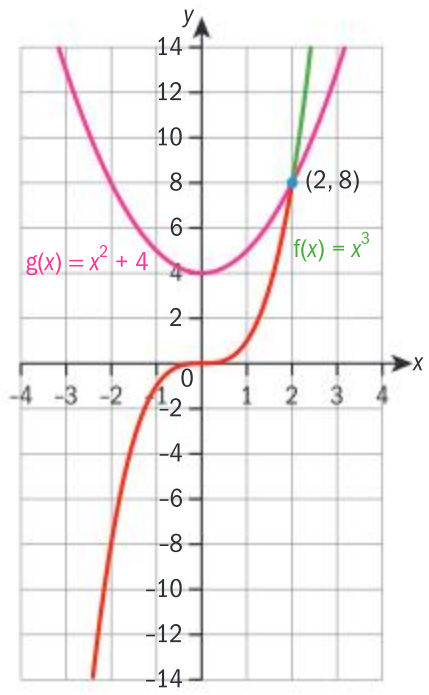
Method 2 is more suitable because you don't need to find the points of intersection but only find zeros, which are always on the x-axis. This saves time as you don't need to change scales in the window, etc.

Sometimes polynomial equations or inequalities can be easily broken to simpler polynomials that you can sketch and find the solution by inspection.



Example 36

Use a simple polynomial graph to solve the inequality $x^3 - x^2 - 4 \leq 0$.

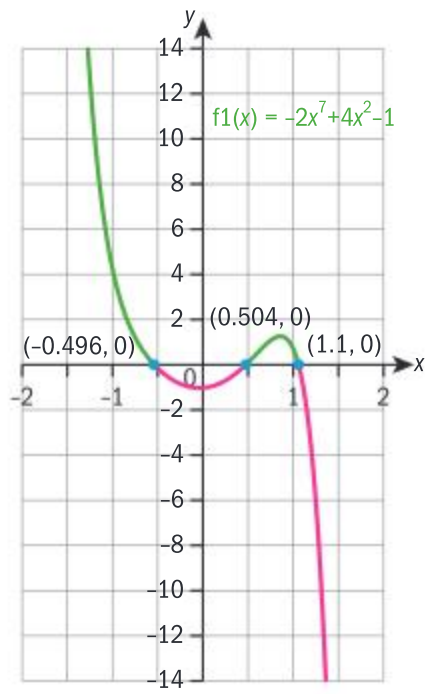
$x^3 - x^2 - 4 \leq 0 \Rightarrow \underbrace{x^3}_{f(x)} \leq \underbrace{x^2 + 4}_{g(x)}$	Rewrite the inequality by using two simple polynomials.
	Or
	Simply enter the original inequality into the GDC.
	Graph those two simple polynomials and find by inspection the point of intersection.
$x \in \mathbb{R}, x \leq 2$	Identify the interval where the graph of f is below the graph of g.

Notice that an equivalent inequality $x^3 - 4 \leq x^2$ would also be an example of simple polynomials to graph and find the solution.

Algebraic methods for solving polynomials are restricted to the third degree in this course. A GDC may be used to solve polynomial equations and inequalities of a higher degree.

Example 37

Use a GDC to solve the inequality $4x^2 < 2x^7 + 1$.

$-2x^7 + 4x^2 - 1 < 0$	Rewrite the inequality as one polynomial.
	
$-0.496 < x < 0.504$ or $x > 1.10$	Graph the polynomial and find the zeros. Identify which part of the graph is below the x-axis.

Notice that the equivalent inequality $2x^7 - 4x^2 + 1 > 0$ gives the same result.

Exercise 30

1 Solve the following inequalities in the set of real numbers by the method of your choice.

a $x^3 + x^2 - 4x - 4 < 0$

b $x^3 + 2x^2 - 9x - 18 > 0$

c $x^3 - 3x^2 + 3x - 2 \leq 0$

d $4x^3 + 8x^2 + x - 3 \quad 0$

e $3x^3 + 4x^2 + 7x + 2 > 0$

f $12x^3 - 16x^2 - 81x - 35 < 0$

g $x^4 + x^3 - 3x^2 - x + 2 \quad 0$

h $2x^4 - x^3 + x^2 - x - 1 < 0$

2 Given the polynomials $f(x) = 3x^3 + 5x^2 - 6x - 9$ and $g(x) = x^3 + 2x^2 - 4x - 4$, find all the values of x such that $f(x) \leq g(x)$.

3 Use simple polynomial graphs to find the solutions of the following inequalities.

a $x^3 + x + 2 > 0$ **b** $-2x^3 + x^2 + 1 \geq 0$

c $x^4 + x^2 - 2 \leq 0$

4 Use a calculator to solve the following inequalities.

a $x^5 - 4x^3 + 2x + 1 \geq 0$

b $x^{13} + 4 < 3x^8 + 5x$

c $x^{15} + 2x^{14} + 5x^8 > 4x^2 - 1$

3.6 Solving systems of linear equations

Systems of two linear equations with two unknowns

Up to this point in your studies, you have only been asked to solve systems of linear equations (such as a pair of simultaneous equations) where there is a unique solution: a single pair of real numbers x and y that satisfy both equations.

In this section, you will learn about the possible types of solutions that can occur when solving systems of linear equations.

Investigation 16

1 Without using a calculator, solve the following simultaneous equations.

a $\begin{cases} 4x + 3y = 18 \\ 7x - 4y = 13 \end{cases}$ **b** $\begin{cases} 2x - 5y = 4 \\ -6x + 15y = 3 \end{cases}$ **c** $\begin{cases} 10x - 4y = 3 \\ -2x + \frac{4}{5}y = -\frac{3}{5} \end{cases}$

2 For each set of equations **a–c**, sketch both lines on the same axes. Where possible, show on your sketch the solution to the pair of equations.

3 **Factual** What are the different types of solution to a system of two linear equations with two unknowns?

4 **Conceptual** Explain the geometrical significance of each type.



When solving systems of two linear equations with two unknowns there are three possible types of solutions that can occur. Each type has a different geometrical interpretation.

1. When there is **a unique pair** of numbers that satisfy both equations, the lines intersect at one point.
2. When there are **no real numbers** that satisfy both equations, the lines are parallel and distinct.
3. When there are **infinitely many pairs** of real numbers that satisfy both equations, the lines coincide.

TOK

If we can find solutions in higher dimensions can we reason that these spaces exist beyond our sense of perception?

All the results of solving systems of linear equations can be checked on a calculator. Make sure you know how to do this using your GDC.



HINT

In type 3, infinitely many pairs of real numbers (x, y) satisfy this system where y can be **any real value**, whilst x is a value **dependent** on y. Instead of expressing y in terms of x as usual, a calculator may provide an answer where x is expressed in terms of y.

Example 38

Find the value of a real parameter m such that the system

$\begin{cases} 4x - y = 1 \\ 3x + my = 2 \end{cases}$ has no solution.

$\begin{cases} 4x - 1 = y \\ 3x + m(4x - 1) = 2 \end{cases} \Rightarrow \begin{cases} 4x - 1 = y \\ (3 + 4m)x = 2 + m \end{cases}$ $x = \dfrac{2 + m}{3 + 4m}$ $3 + 4m = 0 \Rightarrow m = -\dfrac{3}{4}$	Apply the method of substitution by expressing y as the subject in the first equation, and substituting it in the second equation. The system has no solution when $3 + 4m = 0$, since then you would be dividing by zero.

The system of equations in Example 38 can either have no solutions (in the case where $m = -\frac{3}{4}$), or a unique solution (in the case where $m \neq -\frac{3}{4}$).

It is not possible for this particular system to have infinitely many solutions since the coefficients of the variable x and the constant coefficients are not proportional, ie $\frac{4}{3} \neq \frac{1}{2}$.

TOK

How accurate is a visual representation of a mathematical concept?

Example 39

Consider the system of equations $\begin{cases} 2x + (p-1)y = 1 \\ px + 3y = 2 \end{cases}$.

a Find the values of the real parameter p such that the system has a unique solution.

b Find the unique solution in terms of p.

a $\begin{cases} 2x + (p-1)y = 1 \mid \times p,\ p \neq 0 \\ px + 3y = 2 \mid \times 2 \end{cases}$	Use the method of elimination; multiply the first equation by p and the second by 2.
$\Rightarrow \begin{cases} 2px + p(p-1)y = p \\ 2px + 6y = 4 \end{cases} \Rightarrow (p^2 - p - 6)y = p - 4$	Subtract the two equations to eliminate x.
$\Rightarrow p^2 - p - 6 \neq 0 \Rightarrow (p-3)(p+2) \neq 0$ $p \neq 3$ and $p \neq -2$	In order to have a unique solution the coefficient next to y must not be equal to 0.
b $(p^2 - p - 6)y = p - 4 \Rightarrow y = \frac{p-4}{p^2 - p - 6}$	Use the elimination in part **a** to express y in terms of p.
$px = -3y + 2 \Rightarrow px = \frac{-3(p-4)}{p^2 - p - 6} + 2$	Substitute your value of y into the second equation.
$\Rightarrow px = \frac{-3p + 12 + 2p^2 - 2p - 12}{p^2 - p - 6}$	Solve to find x in terms of p.
$\Rightarrow x = \frac{2p^2 - 5p}{p(p^2 - p - 6)} \Rightarrow x = \frac{2p - 5}{p^2 - p - 6}$	
$(x, y) = \left(\frac{2p-5}{p^2 - p - 6}, \frac{p-4}{p^2 - p - 6} \right),\ p \neq -2, 3$	

Notice that the case when $p = 0$ is covered by the unique solution $(x, y) = \left(\frac{5}{6}, \frac{2}{3}\right)$.



Exercise 3P

1 In each case, find the value of the real parameter m such that the system of equations has no solution.

a $\begin{cases} mx + 6y = -2 \\ 7x + 3y = 5 \end{cases}$ **b** $\begin{cases} (m-1)x + 2y = -1 \\ 3x + my = 11 \end{cases}$

2 In each case, find the values of the real parameter p such that the system of equations has infinitely many solutions. Find the solutions.

a $\begin{cases} 6x + py = 8 \\ 3x + y = 4 \end{cases}$

b $\begin{cases} px - y = p \\ (p-4)x + (p-1)y = -2 \end{cases}$

3 In each case, find the values of the real parameter s such that the system of equations has a unique solution. Hence find the solution in terms of s.

a $\begin{cases} 2x - sy = 1 \\ 3x - y = 2 \end{cases}$ **b** $\begin{cases} (s+2)x + y = s \\ (5-2s)x + sy = 4 \end{cases}$

4 The system

$$\begin{cases} a(x+y) + b(x-y) = 1 \\ a(x-y) + b(x+y) = 1 \end{cases}, a, b \in \mathbb{R}, a, b \neq 0$$

is given.

a Find the solution (x, y) in terms of a and b.

b State and explain whether it is possible for the system to have no solution.

Now let's take two linear equations in a general form and use the method of elimination to find a general solution.

Example 40

Find the general formula for a unique solution of $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}, a, b, c, d \neq 0.$

$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \Rightarrow \begin{cases} adx + bdy = ed \\ bcx + bdy = fb \end{cases} -$	Multiply the first equation by d and the second equation by b to obtain equal coefficients of the variable y.
$\Rightarrow adx - bcx \Rightarrow ed - fb$	Subtract the equations to eliminate the variable y.
$\Rightarrow x(ad - bc) \Rightarrow ed - fb$	Use distribution to make x the subject.
$\Rightarrow x = \frac{ed - fb}{ad - bc}$	
$a \times \frac{ed - fb}{ad - bc} + by = e$	Substitute x in one of the equations, eg the first equation.

Continued on next page

$\Rightarrow by = \dfrac{e(ad-bc)-a(ed-fb)}{ad-bc}$	
$= \dfrac{ead-ebc-aed+afb}{ad-bc}$	
$= \dfrac{afb-ebc}{ad-bc}$	
$\Rightarrow y = \dfrac{b(af-ec)}{ad-bc} \times \dfrac{1}{b}$	Make y the subject.
$\Rightarrow y = \dfrac{af-ec}{ad-bc}.$	
$\Rightarrow (x,y) = \left(\dfrac{ed-fb}{ad-bc}, \dfrac{af-ec}{ad-bc}\right), ad-bc \neq 0$	State the condition that the denominator cannot be equal to 0, which is required for the system to have a unique solution.

In Example 40 you could have first eliminated x instead of y, which would have given the same end result.

Did you know?

Gabriel Cramer (1704–1752) has developed this formula within his work on determinants. The form $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a determinant. It is a numerical value such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

All the expressions in the values of x and y in Example 40 can be written as determinants in the following way.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = D, \quad \begin{vmatrix} e & b \\ f & d \end{vmatrix} = D_x, \quad \begin{vmatrix} a & e \\ c & f \end{vmatrix} = D_y$$

Hence, to calculate the values of x and y we need to calculate those three determinants and then we use the formulas $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, $D \neq 0$.

Using determinants to calculate unique solutions works very efficiently when the coefficients of the simultaneous linear equations are complex numbers.



Example 41

Use the formula from Example 40 to solve these simultaneous equations with complex coefficients.

$$\begin{cases} 3x + 2\mathrm{i}y = 5 + 7\mathrm{i} \\ (1-\mathrm{i})x - y = \mathrm{i} \end{cases}$$

Check your answers using a calculator.

$a = 3,\ b = 2\mathrm{i},\ c = 1 - \mathrm{i},\ d = -1,\ e = 5 + 7\mathrm{i},\ f = \mathrm{i}$	Identify the coefficients.
$3\times(-1) - 2\mathrm{i}\times(1 - \mathrm{i}) = -3 - 2\mathrm{i} - 2 = -5 - 2\mathrm{i}$	Using your result from Example 40, the denominator is $ad - bc$.
$(5 + 7\mathrm{i})\times(-1) - 2\mathrm{i} \times \mathrm{i} = -5 - 7\mathrm{i} + 2 = -3 - 7\mathrm{i}$	The numerator for x is $ed - fb$.
$3\times\mathrm{i} - (5 + 7\mathrm{i})\times(1 - \mathrm{i}) = 3\mathrm{i} - 5 + 5\mathrm{i} - 7\mathrm{i} - 7 = -12 - \mathrm{i}$	The numerator for y is $af - ec$.
$x = \dfrac{-3-7\mathrm{i}}{-5-2\mathrm{i}} \times \dfrac{-5+2\mathrm{i}}{-5+2\mathrm{i}} = \dfrac{15-6\mathrm{i}+35\mathrm{i}+14}{29} = 1+\mathrm{i}$	$x = \dfrac{ed-fb}{ad-bc}$. Divide complex numbers by multiplying top and bottom by the conjugate of the denominator.
$y = \dfrac{-12+\mathrm{i}}{-5-2\mathrm{i}} \times \dfrac{-5+2\mathrm{i}}{-5+2\mathrm{i}} = \dfrac{60-24\mathrm{i}-5\mathrm{i}-2}{29} = 2-\mathrm{i}$	$y = \dfrac{af-ec}{ad-bc}$
A GDC can also be used to confirm the result.	

Exercise 3Q

Solve the following simultaneous equations and check your answers with a GDC.

1 $\begin{cases} 3x + (1+\mathrm{i})y = 4 + 5\mathrm{i} \\ (4-\mathrm{i})x - (1-\mathrm{i})y = 7 \end{cases}$

2 $\begin{cases} (1+4\mathrm{i})x + 3\mathrm{i}y = 2 + 4\mathrm{i} \\ (3-5\mathrm{i})x + (5+4\mathrm{i})y = 21 - 27\mathrm{i} \end{cases}$

Systems of three linear equations with three unknowns

When you are presented with a system of three equations with three unknowns, you have to reduce them to a system of two equations with two unknowns, which you know how to solve. In this section, you will focus on solving systems algebraically, either by substitution or elimination.

Example 42

Solve the simultaneous equations $\begin{cases} 2x+y-z=5 \\ x-2y-3z=0 \\ 5x-y+2z=1 \end{cases}$ by

a the method of substitution

b the method of elimination.

a $x=2y+3z \Rightarrow \begin{cases} 2(2y+3z)+y-z=5 \\ 5(2y+3z)-y+2z=1 \end{cases}$

$\Rightarrow \begin{cases} 5y+5z=5 \\ 9y+17z=1 \end{cases} \Rightarrow \begin{cases} y+z=1 \\ 9y+17z=1 \end{cases}$

Isolate one of the variables from one of the equations and substitute it in the remaining two equations. Here we find x in the second equation. Try to target the simplest possible expression.

$y=1-z \Rightarrow 9(1-z)+17z=1 \Rightarrow 8z=-8$

$\Rightarrow z=-1 \Rightarrow y=2 \Rightarrow x=1$

$\Rightarrow (x, y, z)=(1, 2, -1)$

You can apply the method of substitution again, by expressing y in terms of z, or vice versa.

Find z and then the values of y and x respectively.

b

$$\left.\begin{matrix} 2x+y-z=5 \\ 5x-y+2z=1 \end{matrix}\right] + \left.\begin{matrix} \\ \end{matrix}\right\} \Rightarrow \begin{cases} 7x+z=6 \\ 5x-5z=10 \end{cases}$$

$$\left.\begin{matrix} 4x+2y-2z=10 \\ x-2y-3z=0 \end{matrix}\right] + \left.\begin{matrix} \\ \end{matrix}\right\}$$

Add equations 1 and 3.

Add 2 × equation 1 to equation 2.

This eliminates y to give two equations in two unknowns.

When eliminating a variable you must eliminate the same variable, and you must use a combination of all three equations in the process.

$$\Rightarrow \left\{\begin{matrix} 7x+z=6 \\ x-z=2 \end{matrix}\right] + \Rightarrow 8x=8 \Rightarrow x=1$$

Add the two equations to eliminate z. Find the value of x.

$\Rightarrow 7+z=6 \Rightarrow z=-1$

$\Rightarrow 2+y-(-1)=5 \Rightarrow y=2$

$\Rightarrow (x, y, z)=(1, 2, -1)$

You can again use a calculator to check your solution.

Use this x to find z, and then use x and z to find y.



Notice that once you obtain the system of two equations with two unknowns, you don't have to apply the same method again to solve this system. If you used elimination to find a system of two equations with two unknowns, you could then use substitution to solve them, or vice versa.

When solving equations you can notice that you are just operating on the variables' coefficients and the free coefficients (independent of the variables). Instead of always rewriting everything with the variables you can just organize the coefficients in rows and calculate the elementary operations to obtain the result.

$\begin{cases} 2 & 1 & -1 & 5 \\ 1 & -2 & -3 & 0 \\ 5 & -1 & 2 & 1 \end{cases} \Rightarrow \begin{cases} 1 & -2 & -3 & 0 \\ 2 & 1 & -1 & 5 \\ 5 & -1 & 2 & 1 \end{cases}$	Swap the first two rows to simplify the calculations.
$\Rightarrow \begin{cases} 1 & -2 & -3 & 0 \\ 0 & 5 & 5 & 5 \\ 0 & 9 & 17 & 1 \end{cases} \Rightarrow \begin{cases} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 9 & 17 & 1 \end{cases}$	Multiply the first row by 2 and subtract it from the second row. Multiply the first row by 5 and subtract it from the third row.
$\Rightarrow \begin{cases} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 8 & -8 \end{cases} \Rightarrow \begin{cases} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{cases}$	Divide the second row by 5 to simplify the calculations. Multiply the second row by 9 and subtract it from the third row.
$\Rightarrow \begin{cases} x - 2y - 3z = 0 \\ y + (-1) = 1 \\ z = -1 \end{cases} \Rightarrow \begin{cases} x - 2y - 3z = 0 \\ y = 2 \\ z = -1 \end{cases}$	Divide the third row by 8 to simplify the calculations.
$\Rightarrow \begin{cases} x - 2 \times 2 - 3(-1) = 0 \\ y = 2 \\ z = -1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = -1 \end{cases}$	Now rewrite the coefficients into the equations and find the solutions from the third equation up.

This method is called row reduction or the Gaussian method and it is generally used for matrices.

Investigation 17

1 Use your calculator to solve the following systems of simultaneous equations.

a $\begin{cases} 2x + y - z = 5 \\ x - 2y - 3z = 0 \\ 5x - y + 2z = 1 \end{cases}$

b $\begin{cases} 3x + y - 2z = 4 \\ 5x - y + z = -3 \\ 6x + 2y - 4z = 5 \end{cases}$

Continued on next page

c $\begin{cases} x-2y+3z=4 \\ 2x-4y+6z=8 \\ -3x+6y-9z=-12 \end{cases}$

d $\begin{cases} 7x+2y+z=3 \\ 11x-10y+2z=1 \\ 23x+18y+3z=-3 \end{cases}$

e $\begin{cases} x+y-2z=-12 \\ 3x-3y+2z=22 \\ 7x-2y-2z=3 \end{cases}$

f $\begin{cases} 6x+15y-21z=4 \\ 8x+2y-11z=15 \\ -2x-5y+7z=17 \end{cases}$

2 **Conceptual** How can you classify the types of solutions to systems of three linear equations with three unknowns?

3 **Factual** What is the difference between the sets of infinitely many solutions to the equations in parts **c** and **e**?

4 **Factual** What is the relationship between the coefficients of the three equations in part **c**?

When solving systems of three simultaneous linear equations with three unknowns there are again three possible types of solution that can occur.

1. There is **a unique triplet** of numbers that satisfy all three equations.
2. There is **no triplet** of real numbers that satisfy all the equations.
3. There are **infinitely many triplets** of real numbers that satisfy all the equations.

Example 43

Discuss all possible types of solution of the system $\begin{cases} x-y+mz=2 \\ 2x+y-z=3 \\ x+y-2z=5 \end{cases}$ with respect to the real parameter m.

$\begin{cases} x-y+mz=2 \\ 2x+y-z=3 \\ x+y-2z=5 \end{cases} \Rightarrow \begin{cases} 2x+(m-2)z=7 \\ x+z=-2 \end{cases}$	Eliminate the variable y by adding the first and third equations and subtracting the second and third equations.
$\Rightarrow \begin{cases} 2x+(m-2)z=7 \\ 2x+2z=-4 \end{cases} \begin{matrix} \\ - \end{matrix} \Rightarrow (m-4)z=11$	Eliminate x by multiplying the second equation by 2 and subtracting it from the first equation.



$$\Rightarrow z = \frac{11}{m-4},\ m \neq 4$$

$$\Rightarrow x = -2 - \frac{11}{m-4} = \frac{-2m-3}{m-4}$$

$$\Rightarrow y = 5 - \left(\frac{-2m-3}{m-4}\right) + 2\left(\frac{11}{m-4}\right) = \frac{7m+5}{m-4}$$

When $m \neq 4$ there is a unique solution

$$\Rightarrow (x, y, z) = \left(\frac{-2m-3}{m-4}, \frac{7m+5}{m-4}, \frac{11}{m-4}\right),\ m \neq 4$$

$m = 4 \Rightarrow 0 \times z = 11 \Rightarrow 0 = 11$, which is false.

When $m = 4$ the system has no solution.

If a real solution exists, you need to divide by the expression $(m - 4)$ which cannot be equal to 0.

Find x in terms of m.

Find y in terms of m.

When $m = 4$ from the equation you obtain a **false** statement, therefore the system has no solution.

Determining the equation of a polynomial from a system of linear equations

Suppose you are told the coordinates of two points which lie on the graph of the linear function $f(x) = ax + b$. You can use these points to formulate two equations in two unknowns, a and b. By solving this system of equations, you can determine the equation of the function.

In a similar way, when you know three points through which the graph of a quadratic function $f(x) = ax^2 + bx + c$ passes, you can formulate three equations in three unknowns a, b and c. Solving this system of equations gives the equation of the quadratic.

Example 44

Find the equation of the quadratic function $f(x) = ax^2 + bx + c$ that passes through the points $(-1, 10)$, $(2, -2)$ and $(4, 0)$.

$$(-1, 10): a(-1)^2 + b(-1) + c = 10$$
$$\Rightarrow a - b + c = 10 \quad (1)$$

$$(2, -2): a \times 2^2 + b \times 2 + c = -2$$
$$\Rightarrow 4a + 2b + c = -2 \quad (2)$$

$$(4, 0): a \times 4^2 + b \times 4 + c = 0$$
$$\Rightarrow 16a + 4b + c = 0 \quad (3)$$

Input the coordinates of each point on the parabola into the equation.

Continued on next page

$(2)-(1)$ $(3)-(1)$ $\Rightarrow \begin{cases} 3a+3b=-12 \\ 15a+5b=-10 \end{cases} \Rightarrow \begin{cases} a+b=-4 \\ 3a+b=-2 \end{cases}$	Eliminate variable c by subtracting the first equation from the second and third equations.
$2a=2 \Rightarrow a=1 \Rightarrow 1+b=-4 \Rightarrow b=-5$	Subtract the first equation from the second to eliminate b.
$1+5+c=10 \Rightarrow c=4$	Find a, b and c.
$\Rightarrow f(x)=x^2-5x+4$	Write the equation of the function using your values of a, b and c.

HINT

You can use your GDC in two different ways to solve this type of problem.

- You can formulate the three linear equations and use your GDC to solve them.
- You can use the statistical feature for finding the regression equation of a quadratic curve which passes through three specified points.

Exercise 3R

1 Find the value(s) of the real parameter m so that the system has no unique solution.

a $\begin{cases} 2x+y-z=4 \\ x-y+mz=2 \\ 3x+2y-z=1 \end{cases}$ **b** $\begin{cases} x+my+z=1 \\ 3x+y+mz=-2 \\ x+2y+z=3 \end{cases}$

2 Find the value(s) of the real parameter k so that the system has infinitely many solutions. Find the solutions.

a $\begin{cases} 2x+3y-z=1 \\ kx+9y-3z=3 \\ 3x+7y+5z=2 \end{cases}$

b $\begin{cases} x+ky+z=0 \\ 4x+6y+(k+2)z=-6 \\ 2x+3y+2z=-3 \end{cases}$

3 Find the values of the real parameter m so that the system $\begin{cases} 2x+y+z=m \\ x+my+z=m+1 \\ x+y+mz=-3 \end{cases}$ has a unique solution.

4 Find the equation of a quadratic function $f(x)=ax^2+bx+c$ that passes through the points

a $(-3, 1)$, $(-2, -5)$ and $(1, 4)$

b $(-1, 1)$, $(1, -9)$ and $(2, 8)$

Developing your toolkit

Now do the Modelling and investigation activity on page 216.



Chapter summary

- **The quadratic formula**

  For a quadratic equation in the form $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$, the solutions or roots are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- Given a quadratic equation of the form $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$, the **discriminant** is the expression in the formula that is under the square root and is denoted by Greek letter Δ, $\Delta = b^2 - 4ac$.
  - **Case 1:** $\Delta > 0$

    If the discriminant is positive, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and there are **two distinct real roots**.
  - **Case 2:** $\Delta = 0$

    If the discriminant is equal to zero, then $x = \dfrac{-b}{2a}$. This is regarded as **one repeated root**.
  - **Case 3:** $\Delta < 0$

    If the discriminant is less than zero, then $\sqrt{b^2 - 4ac}$ is not real. In this case, there is **no real solution**.
- Complex numbers are numbers of the form $z = a + \mathrm{i}b$, where $a, b \in \mathbb{R}$.

  a is called the **real part** of z, and we write $\mathrm{Re}(z) = a$.

  b is called the **imaginary part** of z, and we write $\mathrm{Im}(z) = b$.

  The set of complex numbers is denoted by $\mathbb{C}$.
- Given the complex number $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, the **modulus** of z is given by

  $|z| = |x + \mathrm{i}y| = \sqrt{x^2 + y^2} = \sqrt{\left(\mathrm{Re}(z)\right)^2 + \left(\mathrm{Im}(z)\right)^2}$.
- Two complex numbers $z_1 = a + b\mathrm{i}$ and $z_2 = c + d\mathrm{i}$, $a, b, c, d \in \mathbb{R}$, are equal if and only if their real parts are equal and their imaginary parts are equal, $a = c$ and $b = d$.
- Addition and multiplication by a scalar:

  $z_1 + z_2 = (a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}, \quad a, b, c, d \in \mathbb{R}$

  $\lambda z = \lambda(a + b\mathrm{i}) = (\lambda a) + (\lambda b)\mathrm{i}, \quad \lambda, a, b \in \mathbb{R}$
- Multiplication of complex numbers:

  $z_1 \times z_2 = (a + b\mathrm{i}) \times (c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}, \quad a, b, c, d \in \mathbb{R}$
- For any complex number $z = a + b\mathrm{i}$, $a, b \in \mathbb{R}$, there is a conjugate complex number of the form $z^* = a - b\mathrm{i}$. Their real parts are equal, $\mathrm{Re}(z) = \mathrm{Re}(z^*)$, and their imaginary parts are opposite, $\mathrm{Im}(z) = -\mathrm{Im}(z^*)$.
- Division of complex numbers:

  $\dfrac{z_1}{z_2} = \dfrac{z_1 \times z_2^*}{|z_2|^2}$
- Powers of i: $\mathrm{i}^n = \begin{cases} 1, & n = 4k \\ \mathrm{i}, & n = 4k + 1 \\ -1, & n = 4k + 2 \\ -\mathrm{i}, & n = 4k + 3 \end{cases}, k \in \mathbb{Z}$

Continued on next page

- Polynomials are functions which map a real variable, often called x, to another real number. We write f: $\mathbb{R} \to \mathbb{R}$.
- Polynomials are functions of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_i \in \mathbb{R}$, $i = 0, \ldots, n$ are called the **coefficients.** The highest power (n) of the variable x is called the **degree** of the polynomial, and we write $\deg(f) = n$.
- A **linear combination** of two functions f and g is an expression of the form $a \times f(x) + b \times g(x)$, where a and b are real numbers.

  A linear combination of n functions is an expression of the form $\sum_{i=1}^{n} a_i \times f_i(x)$, where f_i are functions and $a_i \in \mathbb{R}$.

- Two polynomials $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$, and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_2 x^2 + b_1 x + b_0$, $b_m \neq 0$, are said to be **equal** if and only if:
  - they have the **same degree**, $n = m$
  - all the **corresponding coefficients** are **equal**: $a_i = b_i$ for all $i = 0, 1, \ldots, n$.
- For any two polynomials f and g there are unique polynomials q and r such that $f(x) = g(x) \times q(x) + r(x)$ for all real values of x.

  The polynomial q is called the **quotient** and the polynomial r is called the **remainder**. Notice that $\deg(g) > \deg(r)$.
- **Polynomial remainder theorem**

  Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$, $a_n \neq 0$, and a real number p, then the remainder when $f(x)$ is divided by the linear expression $(x - p)$ is $f(p)$.
- **Factor theorem**

  A polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n$, $a_n \neq 0$, has a factor $(x - p)$, $p \in \mathbb{R}$ if and only if $f(p) = 0$.
  - **Corollary**

    Given a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, $a_n \neq 0$, and real numbers a and b, $a \neq 0$, then the remainder when $f(x)$ is divided by the linear expression $(ax - b)$ is $f\left(\frac{b}{a}\right)$.
- **The fundamental theorem of algebra (FTA)**

  A polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$, with real or complex coefficients has at least one zero. There is an $\omega \in \mathbb{C}$ such that $f(\omega) = 0$.
- **Lemma**

  Each polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$, with real coefficients can be written in a factor form $f(x) = a_n(x - \omega_1)(x - \omega_2)\ldots(x - \omega_n)$ such that $\omega_k \in \mathbb{C}$, $k = 1, \ldots, n$.



- **Conjugate root theorem**

  Given a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$, $a_n \neq 0$, that has a complex zero z, then its conjugate z^* is also a zero of the polynomial f.

- If a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with real or complex coefficients $(a_n \neq 0)$ has zeros $x_1, x_2, \ldots, x_n$ then $\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \left(x_{i_1} x_{i_2} \ldots x_{i_k} \right) = (-1)^k \frac{a_{n-k}}{a_n}$, $1 \leq k \leq n$; specifically, for the sum and the product,

$$\begin{cases} x_1 + x_2 + x_3 + \ldots + x_n = -\dfrac{a_{n-1}}{a_n} \\ x_1 x_2 x_3 \ldots x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

- When solving systems of two linear equations with two unknowns there are three possible types of solutions that can occur. Each type has a different geometrical interpretation.
  - When there is a **unique pair** of numbers that satisfy both equations, the lines intersect at one point.
  - When there are **no real numbers** that satisfy both equations, the lines are parallel and distinct.
  - When there are **infinitely many pairs** of real numbers that satisfy both equations, the lines coincide.
- When solving systems of three simultaneous linear equations with three unknowns there are again three possible types of solutions that can occur.
  - There is a **unique triplet** of numbers that satisfy all three equations.
  - There is **no triplet** of real numbers that satisfy all the equations.
  - There are **infinitely many triplets** of real numbers that satisfy all the equations.
- **Finding polynomials by points**

  When you know three points through which a quadratic function $f(x) = ax^2 + bx + c$ passes, you can formulate three equations in three unknowns a, b and c. Solving these gives the equation of the quadratic.

Developing inquiry skills

Return to the opening problem. Your city council has asked you to prepare a plan of a roller coaster for a new amusement park.

How would you go about this task using what you have learned in this chapter?